

Antenna functions from MHV rules

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ABSTRACT: QCD amplitudes display a universal behaviour when one or more partons are soft and/or collinear. This can be expressed in terms of antenna functions which are much simpler than the full amplitudes and yet correctly embody their infrared behaviour. We show how antenna functions can be naturally obtained via a twistor-inspired MHV approach. As an application, we present compact results for MHV and NMHV antenna functions valid for any number of gluons. These are sufficient to calculate the complete set of tree-level gluon antenna functions up to N³LO. As an interesting corollary, we prove that splitting amplitudes too can be written directly through a MHV diagrammatic approach. Finally we find that antenna functions, collinear splitting amplitudes and eikonal factors satisfy the same kind of recursive relation as the full amplitudes.

KEYWORDS: QCD, NLO Computations.

Contents

1. Introduction	1
2. Tree-level techniques in QCD	4
3. Infrared factorisation of tree-level QCD amplitudes	6
4. Power counting for antenna functions in the CSW formalism	9
5. Diagrammatic evaluation of antenna functions in the CSW formalism	14
6. Discussion and results	19
7. From antenna functions to splitting amplitudes	22
8. Recursive relations	26
9. Conclusion	28
A. Reconstruction functions	30
B. Gluon insertion rule	31
C. Recursive formulation of the CSW formalism	32
D. Recursive relations for soft factors	37
E. Explicit results	41

1. Introduction

Progress in computations in perturbative QCD has witnessed an impressive acceleration in recent past: calculations that only a few years ago were considered out-of-reach have been completed and many more are now in sight. Particularly impressive are the developments of new methods for computing one-loop amplitudes with many external legs and two-loop amplitudes, that bypass the commonly employed Feynman diagrams techniques and lead to compact expressions and possibly to an efficient automatization of NLO and NNLO computations. Such glaring advances have mainly come from a spur of activity initiated in 2003 by Witten's suggestion of a duality between a string theory in a twistor space and QCD [1]. Starting from the work of Cachazo, Britto et al. [2], and building up from the previous seminal work by Bern, Dixon and Kosower [3, 4], it has been quickly realized that

generalised unitarity methods could provide the philosopher’s stone that, loosely speaking, would turn loop computations into tree-level ones. Since then many new competing and promising methods to evaluate one-loop amplitudes have been proposed, some of which have been successfully implemented in working codes [5, 6]. Progress in two-loop calculations has been somewhat slower, but also steady.

In this quest, it is not only important to have efficient tools for the computations of the necessary ingredients (loop and tree-level amplitudes) but also a general algorithm that allows their suitable combination to evaluate infrared-safe observables, such as jet rates. Several general algorithms for computing observables at NLO are available since many years, the most popular one being the subtraction method introduced by Catani and Seymour [7].

Understanding the infrared singular behaviour of tree-level QCD amplitudes is a prerequisite for computing infrared-finite cross sections at fixed order in perturbation theory. In general, when one or more final state particles are either soft or collinear, the amplitudes factorise into a product of a scattering amplitude that depends only on the remaining hard partons in the process (including any hard partons constructed from an ensemble of unresolved partons) and a splitting amplitude containing all the singularities due to the unresolved particles. This factorisation is universal and can be generalised to more particles [8–12] and any number of loops [13].

Although several NLO implementations of this technique have already been around for many years, at NNLO the infrared structure becomes much more complicated due to an overlapping of the different singular regions of phase space, and a complete and process-independent subtraction scheme is still to be worked out. In this case also, several proposals have been put forward. A first one by Trocsanyi and Somogyi resolves the complicated phase space structure by using a special form of the soft and splitting amplitudes such that there are no overlapping regions by construction [14–18]. A second one is based on antenna functions at the amplitude level [19–21] and amplitude squared level [8, 22], a generalisation of the Catani-Seymour dipoles, which interpolate between the different soft and collinear regions and avoid in this way the double-counting problem. This method is the only one that has been successfully applied in a non-trivial calculation so far, i.e., the NNLO 3-jet rates in e^+e^- collisions [23–25]. Here subtraction terms derived from full matrix elements can be viewed as antenna functions, encapsulating all singular limits due to unresolved partonic emission between two colour-connected hard partons [8, 26]. In particular, process-independent antenna functions describing arbitrary QCD multiparticle processes can be directly related to three-parton matrix elements at NLO (one unresolved parton radiating between two colour-connected hard partons) and four-parton matrix elements at NNLO (two unresolved partons radiating between two colour-connected hard partons) [27, 28].

The main purpose of this work is to show that antenna functions can be efficiently defined and calculated at the amplitude level by means of the twistor CSW (or MHV) rules [29, 30]. We apply the MHV rules to study the singular limits of QCD amplitudes when n gluons become soft and/or collinear, and present an alternative definition for tree-level antenna functions. Our work can be considered an extension of the work by Kosower which is based on the standard recursive Berends-Giele relations [19–21] and

that of Birthwright *et al.* where MHV rules were used to derive multi-collinear limits of amplitudes involving quarks and gluons [31, 32].

In refs. [31, 32] it has been shown that any splitting amplitude can be extracted from a given subset of MHV diagrams (D_i^{MHV}), schematically,

$$\text{Split}(1, \dots, n) = \frac{\lim_{1||\dots||n} \sum_{i \in \mathcal{S}} D_i^{\text{MHV}}|_{(1, \dots, n, a, b, c)}}{\text{Born}(1 + \dots + n, a, b, c)}. \quad (1.1)$$

where \mathcal{S} is a significantly smaller subset of all the possible MHV diagrams contributing to a generic amplitude $A_{n+3}(1, \dots, n, a, b, c)$. In this work we generalise the above approach in various directions. First we prove that a similar result holds for the antenna functions, i.e.,

$$\text{Ant}(\hat{a}, \hat{b} \leftarrow a, 1, \dots, n, b) = \frac{\lim_{1 \sim \dots \sim n} \sum_{i \in \mathcal{A}} D_i^{\text{MHV}}|_{(a, 1, \dots, n, b, c, d)}}{\text{Born}(\hat{a}, \hat{b}, c, d)}. \quad (1.2)$$

where we indicated with \sim the possibility of partons being collinear or soft. Second we are able to prove that the same result can be obtained in a much more direct way, i.e., without the need of taking any limit and ratio but by simply summing over the same subset of MHV amplitudes but calculated at the shifted momenta \hat{a}, \hat{b}

$$\text{Ant}(\hat{a}, \hat{b} \leftarrow a, 1, \dots, n, b) = \sum_{i \in \mathcal{A}} D_i^{\text{MHV}}|_{(\hat{a}, \hat{b}, a, 1, \dots, n, b)}. \quad (1.3)$$

Employing such a diagrammatic approach, very compact general formulas for MHV and NMHV antenna functions are obtained. As a corollary we also prove that splitting amplitudes can be built diagrammatically through a similar formula, which manifestly shows the pole structure of the splitting amplitudes.

Eq. (1.3) is the main result of this work. We then proceed by following the same approach presented in ref. [33] to rewrite the usual CSW rules into a recursive form and we are able to recast eq. (1.3) in a recursive form. As a result, we find that the full amplitudes, the antenna functions and the splitting amplitudes obey formally identical relations, the only differences being in the definition of vertices and in the initial conditions. This is our second main result.

The paper is organised as follows. We start reviewing the basic concepts and establish the notation for helicity amplitudes in section 2 and for their infrared factorisation properties in section 3. In section 4 we prove that a power counting argument similar to that employed in refs. [31, 32] for the splitting amplitudes can be easily extended to antenna functions and allows the identification of the set of MHV diagrams entering in eq. (1.2). In the following sections we first prove eq. (1.3), i.e., that antenna functions can be directly calculated by summing over a well-defined class of MHV diagrams and then provide some closed-form results for MHV and NMHV antenna functions. In section 5 we give a simple argument that proves that also splitting amplitudes can be directly built via a diagrammatic approach. In section 8 recursive formulations for both antenna functions and splitting amplitudes are proposed and proved. We summarize our findings in the Conclusion and discuss some open issues. The appendices contain some complementary information, including proofs and explicit results for up to N³LO antennas.

2. Tree-level techniques in QCD

In this section we briefly review the notion of color decomposition of tree-level QCD amplitudes and spinor-helicity formalism. This allows us to present our conventions and to also underline the aspects that will play an important role in the following.

The basic idea of a color decomposition is to factorise the information on the gauge structure from the kinematics. As an example, consider the amplitude for n gluons of colors a_1, a_2, \dots, a_n with $a_i = 1, \dots, N^2 - 1$. One can easily prove that at tree level, such an amplitude can be decomposed as [34]

$$\mathcal{A}_n(\{p_i, h_i, a_i\}) = g^{n-2} \sum_{\sigma \in S_{n-1}} \text{Tr}(T^{a_1} \dots T^{a_{\sigma(n)}}) A_n(1^{h_1}, \dots, \sigma(n^{h_n})), \quad (2.1)$$

where T^a are the fundamental-representation matrices of $SU(N)$, and the sum is over all $(n-1)!$ permutations of $(2, \dots, n)$.¹ Each trace corresponds to a particular color structure. The factor associated with each color structure, A , is gauge invariant and is called a color-ordered amplitude.² It depends on the four-momenta p_i and polarization vectors ϵ_i of the n gluons, represented simply by i in its argument. The color-ordered amplitudes are far simpler to calculate than the full amplitude \mathcal{A} due to the smaller number of Feynman diagrams contributing to them.

As the partial amplitudes are functions only of the helicities and the momenta of the particles, they can be most easily calculated using a spinor-helicity formalism. All partial amplitudes can then be expressed as rational functions of spinor products, defined by

$$\langle ij \rangle \equiv \overline{u_-(k_i)} u_+(k_j) \quad [ij] \equiv \overline{u_+(k_i)} u_-(k_j), \quad (2.2)$$

where $u_{\pm}(k)$ are the helicity-projected solutions of the massless Dirac equation in momentum space. The spinor products fulfill several useful properties among which we recall the Schouten-identity

$$\langle ij \rangle \langle kl \rangle = \langle ik \rangle \langle jl \rangle + \langle il \rangle \langle kj \rangle. \quad (2.3)$$

Although the calculation of the partial amplitudes may in general be a very hard task, there are special classes of partial amplitudes for which the result can be written in a very compact form:

1. Using the effective supersymmetry of QCD at high energy, it is possible to derive supersymmetric Ward identities for tree-level QCD amplitudes. These identities imply that the partial amplitudes where all the particles have the same helicity or only one particle has a different helicity vanish,

$$\begin{aligned} A_n(1^{\pm}, 2^+, \dots, n^+) &= 0, \\ A_n(1^{\mp}, 2^-, \dots, n^-) &= 0. \end{aligned} \quad (2.4)$$

Note that the above relations hold at any order in perturbation theory in supersymmetric gauge theories.

¹For alternative color decompositions see refs. [35, 36].

²Also referred to as a dual amplitude or partial amplitude.

2. The first class of non vanishing partial amplitudes are those with exactly two negative helicity gluons (independently of their position). These amplitudes are the so-called maximally helicity violating (MHV) amplitudes, and were first conjectured by Parke and Taylor in ref. [37], and later proven by Berends and Giele using their recursive relations [38]. MHV-amplitudes at tree-level have a very simple analytic structure given by

$$A_n(1^+, \dots, i^-, \dots, j^-, \dots, n^+) = \frac{\langle ij \rangle^4}{\langle 12 \rangle \langle 23 \rangle \dots \langle (n-1)n \rangle \langle n1 \rangle}, \quad (2.5)$$

and using parity

$$A_n(1^-, \dots, i^+, \dots, j^+, \dots, n^-) = (-1)^n \frac{[ij]^4}{[12][23] \dots [(n-1)n][n1]}. \quad (2.6)$$

It can be easily seen that all amplitudes for $n = 4$ and $n = 5$ can be obtained from the above formulas. However, starting from $n = 6$ on, no general formula valid for any helicity configuration is known.

Until a few years ago the two main techniques used to calculate partial amplitudes were the color-ordered Feynman rules [39] and the Berends-Giele recursive relations [38]. In 2003, E. Witten conjectured a duality between QCD and a string theory in twistor space [1]. The two main outcomes of the twistor approach to QCD at tree-level were the so-called BCF recursive relations [40, 41] and the CSW formalism³ [29, 30]. In the following we will recall briefly the CSW formalism.

The CSW formalism was originally introduced as a conjecture in ref. [29] and later proven recursively in ref. [30]. It states that all tree-level color-ordered amplitudes can be built up by connecting MHV-amplitudes by scalar propagators. However, one then needs to consider MHV-amplitudes where some of the external legs are off-shell. It was shown in ref. [29] that for an off-shell line carrying momentum $P_{a\dot{a}}$, one can choose

$$P_{a\dot{a}}\eta^{\dot{a}}, \quad (2.7)$$

where $\eta^{\dot{a}}$ is an arbitrary antiholomorphic spinor. The MHV rules can be used to calculate tree-level color-ordered amplitudes and are very simple: [29]

1. Write down all possible ways to decompose the amplitude into MHV-amplitudes connected by scalar propagators $1/P^2$.
2. Introduce a term $P_{a\dot{a}}\eta^{\dot{a}}$ for each off-shell leg in an MHV-amplitude.

Note that as the vertices in the CSW formalism correspond to MHV-amplitudes, eq. (2.5), they are completely holomorphic objects.⁴ Thus, in the CSW formalism the only possible source of antiholomorphic spinor products in an amplitude are the scalar propagators. This important property will be extensively used in section 4.

³Also known as MHV rules.

⁴This is equivalent to state that as long as the η -dependence is kept general, the antiholomorphic spinor products coming from the off-shell continuation must cancel out in the end.

3. Infrared factorisation of tree-level QCD amplitudes

It is well known that amplitudes with massless particles in the final state exhibit infrared singularities in the limit where a particle becomes soft or where two or more particles become collinear. A precise knowledge of the infrared behavior of the QCD amplitudes is thus needed to handle the infrared divergencies that appear in QCD calculations beyond LO. The infrared factorisation of QCD amplitudes is universal and the divergencies can be described by universal quantities, known as splitting amplitudes and soft factors. The knowledge of these universal functions is at the basis of the subtraction methods, which handle the infrared divergencies and allow a numerical implementation [7, 8, 14, 16–18, 42, 43]. In this section we review the different factorisation properties of a QCD gluon amplitude. We start by recalling the soft and collinear limits, and finally we also review the antenna factorisation, which provides a tool to describe in a unified way all the infrared singularities contained in an amplitude.

Collinear singularities arise if two or more adjacent particles, say $1, \dots, n$, in a color-ordered amplitude become collinear. In this limit, the amplitude factorises as

$$A_m(1, \dots, n, \dots, m) \sim \sum_h \text{Split}_{-h}(1, \dots, n) A_{m-n+1}(P^h, n+1, \dots, m), \quad (3.1)$$

where P denotes the combined momentum $p_1 + \dots + p_n$, i.e. h denotes the collinear direction. The factorisation property (3.1) implies that in the region of phase space where the particles $1, \dots, n$ become collinear, the amplitude $A_m(1, \dots, n, \dots, m)$ can be approximated by the right-hand side of eq. (3.1).

In the calculation of the splitting amplitudes, one is generally interested to have the simplest of all cases $m - n + 1 = 4$, i.e. $n = 3$, the factorisation over the four-point amplitude,

$$A_{n+3}(1, \dots, n, a^+, b^{-h}, c^-) \sim \text{Split}_{-h}(1, \dots, n) A_4(P^h, a^+, b^{-h}, c^-). \quad (3.2)$$

The first important observation is that, in this case, due to the supersymmetric Ward identities (2.4), no sum over the helicities of the intermediate particle has to be carried out. Furthermore, from eq. (2.5) the hard four-point amplitude always has the simple analytic structure of an MHV-amplitude. The splitting amplitudes can therefore be extracted from an $(n + 3)$ -point amplitude using a simple algorithm, based on a simple power counting:

1. If i and j are particles from the collinear set, rescale the corresponding spinor products as

$$\begin{aligned} \langle ij \rangle &\rightarrow t \langle ij \rangle, \\ [ij] &\rightarrow t [ij]. \end{aligned} \quad (3.3)$$

2. Expand the amplitude in powers of t ,

$$A_{n+3}(1, \dots, n, a^+, b^{-h}, c^-) = \mathcal{O}(1/t^{n-1}) + \dots, \quad (3.4)$$

where the dots indicate terms that are less divergent than $1/t^{n-1}$. The splitting amplitude corresponds to the term of maximal divergence, i.e. $1/t^{n-1}$, if i is in the collinear

set and a is not in the collinear set, then the splitting amplitude corresponds to the term in $1/t^{n-1}$, with

$$\begin{aligned}\langle ai \rangle &\rightarrow \sqrt{z_i} \langle aP \rangle, \\ [ai] &\rightarrow \sqrt{z_i} [aP],\end{aligned}\tag{3.5}$$

where z_i denote the longitudinal momentum fractions.

The factorisation of the amplitude in the soft limit is very different from the collinear one. Consider a pure gluon color-ordered amplitude at tree-level. This amplitude displays an infrared divergence as some of the gluons become soft, i.e. , some of the gluon energies vanish. Two different cases might occur, the color-connected and the color-disconnected, that need to be considered separately [9].

Consider first the color-connected case where the soft gluons are all adjacent in the color-ordered amplitude, say s_1, \dots, s_m . The amplitude then factorises as

$$A_n(1, \dots, a, s_1, \dots, s_m, b, \dots, n) \sim \text{Soft}(a, s_1, \dots, s_m, b) A_{n-m}(1, \dots, a, b, \dots, n).\tag{3.6}$$

This factorisation property implies that in the region of phase space where the particles s_1, \dots, s_m become soft, the amplitude $A_n(1, \dots, a, s_1, \dots, s_m, b, \dots, n)$ can be approximated by the right-hand side of eq. (3.6). The (color-ordered) soft factor $\text{Soft}(a, s_1, \dots, s_m, b)$ defined in this way depends on the momenta and the helicities of the soft particles, and also on the particles a and b which are next to it in the color-ordered amplitude. The soft factors are, however, independent of the helicities of the particles a and b [39].

The color-disconnected case corresponds to the situation where the soft particles are not all adjacent, but separated by at least one hard gluon. In this case the factorisation is, e.g. for two sets of soft particles,

$$\begin{aligned}A_n(1, \dots, a, s_1, \dots, s_m, b, \dots, c, t_1, \dots, t_l, d, \dots, n) \sim \\ \text{Soft}(a, s_1, \dots, s_m, b) \text{Soft}(c, t_1, \dots, t_l, d) A_{n-m-l}(1, \dots, a, b, \dots, c, d, \dots, n),\end{aligned}\tag{3.7}$$

i.e. , the color-disconnected soft factors are products of the corresponding color-connected soft factors. For the rest of this work we will thus only deal with the color-connected case.

Similar to the case of splitting amplitudes, the calculation of soft factors can be simplified by choosing the hard amplitude to be an MHV-amplitude,

$$A_{n+4}(a^+, 1, \dots, n, b^+, c^-, d^-) \sim \text{Soft}(a, 1, \dots, m, b) A_4(a^+, b^+, c^-, d^-).\tag{3.8}$$

Note that as the soft factor is independent of the helicities of the hard particles a and b , it is unaffected by the helicity assignment in the hard part. The soft factors can be obtained by simple power counting, following a set of simple rules:

1. Rescale all spinor products according to

$$\langle ij \rangle \rightarrow t^2 \langle ij \rangle,$$

$$\begin{aligned}
\langle aj \rangle &\rightarrow t \langle aj \rangle, \\
[ij] &\rightarrow t^2 [ij], \\
[a_j] &\rightarrow t [a_j], \\
s_{ij} &\rightarrow t^4 s_{ij}, \\
s_{aj} &\rightarrow t^2 s_{aj}.
\end{aligned}
\tag{3.9}$$

2. Expand the amplitude in power of t ,

$$A_n(1, \dots, a, s_1, \dots, s_m, b, \dots, n) = \mathcal{O}(1/t^{2m}) + \dots, \tag{3.10}$$

where the dots indicate terms which are less divergent than $1/t^{2m}$. The soft factor then corresponds to the “maximal divergence”, i.e. , to the term in $1/t^{2m}$.

The collinear and soft factors presented above are sufficient to describe all of the infrared divergencies of a QCD amplitude. However, it is sometimes more convenient to define a general class of objects that describe all the infrared singularities in one single function, the so-called antenna function [19–21]. Let us consider an N -point pure gluon amplitude $A_m(a, 1, \dots, n, b, \dots, m)$. The singular limit for the particles $1, \dots, n$ is defined as the limit where the particles $1, \dots, n$ are either collinear to a , collinear to b or soft.

An antenna function is defined as any function $\text{Ant}(\hat{a}^{h_{\hat{a}}}, \hat{b}^{h_{\hat{b}}} \leftarrow a, 1, \dots, n, b)$ that reproduces the correct singular limits. Note that, at variance with the splitting amplitudes and the soft factors, the definition above does not lead to unique functions. In the singular regions of phase space the amplitude can be approximated by

$$A_m(a, 1, \dots, n, b, \dots, m) \sim \sum_{h_{\hat{a}}, h_{\hat{b}}} \text{Ant}(\hat{a}^{h_{\hat{a}}}, \hat{b}^{h_{\hat{b}}} \leftarrow a, 1, \dots, n, b) A_{m-n}(\hat{a}^{-h_{\hat{a}}}, \hat{b}^{-h_{\hat{b}}}, \dots, m), \tag{3.11}$$

where $k_{\hat{a}}$ and $k_{\hat{b}}$ are the so-called reconstruction functions,

$$\begin{aligned}
k_{\hat{a}} &= f_{\hat{a}}(a, 1, \dots, n, b), \\
k_{\hat{b}} &= f_{\hat{b}}(a, 1, \dots, n, b).
\end{aligned}
\tag{3.12}$$

The reconstruction functions satisfy the following properties:

1. On-shellness, $k_{\hat{a}}^2 = k_{\hat{b}}^2 = 0$.
2. Momentum conservation, $k_{\hat{a}} + k_{\hat{b}} = k_a + k_1 + \dots + k_n + k_b$.
3. They reduce to the right expressions in the various singular limits, e.g. if $1, \dots, n$ become collinear to a , then $k_{\hat{a}} \rightarrow k_a + k_1 + \dots + k_n$ and $k_{\hat{b}} \rightarrow k_b$.
4. They leave the leading pole unchanged.

Explicit expressions for the reconstruction functions are collected in appendix A. As we already pointed out, the definition of an antenna function is not strict enough to uniquely fix its functional form and several definitions can be found in the literature. In ref. [20],

Kosower presented a way to build an antenna function, based on the well known Berends-Giele recursive relations. In refs. [8, 27, 28], Gehrmann, Gehrmann-De Ridder and Glover defined antenna functions as ratios between the full and the hard squared matrix elements. In the following, we will present an alternative definition based on the CSW formalism and the properties special to this approach.

4. Power counting for antenna functions in the CSW formalism

The CSW formalism provides a very powerful method to efficiently calculate partial amplitudes. It is then natural to wonder whether it can also be used to calculate splitting amplitudes and/or soft factors. In refs. [31, 32], Birthwright et al. showed that it is indeed possible to derive very simple and compact formulas for several classes of splitting amplitudes, using the fact that in the CSW formalism the pole structure of an amplitude is manifest. They derived rules to identify the CSW diagrams that contribute to a given collinear limit just by looking at the CSW pole structure.

In the following we show that it is possible to generalize the rules of refs. [31, 32] such that they do not only describe the collinear behavior of an amplitude, but the full infrared behavior, i.e. , the antenna functions.

We start by briefly reviewing the calculation of splitting amplitudes using the CSW formalism presented in refs. [31, 32]. The CSW pole structure of splitting amplitudes is given by

$$\begin{aligned} \text{Split}_- &\sim \frac{1}{[\]^{n_-}} f(\langle \ \rangle), \\ \text{Split}_+ &\sim \frac{1}{[\]^{n_- - 1}} f(\langle \ \rangle), \end{aligned} \tag{4.1}$$

where n_- is the number of negative-helicity gluons in the collinear set. In this case, if the hard amplitude is chosen to be a four-point amplitude, eq. (3.2), the CSW diagrams that contribute in the collinear limit are exactly those where all CSW propagators go on-shell. Furthermore, the rules introduced in the previous section, eq. (3.5), in the CSW formalism become

$$\begin{aligned} \langle a P_{i,j} \rangle &\rightarrow \langle a P \rangle [P \eta] \sum_{\alpha=i}^j z_\alpha, \\ \langle k P_{i,j} \rangle &\rightarrow [P \eta] \sum_{\alpha=i}^j \sqrt{z_\alpha} \langle k \alpha \rangle, \\ \langle P_{i,j} P_{k,l} \rangle &\rightarrow [P \eta]^2 \sum_{\alpha=i}^j \sum_{\beta=k}^l \sqrt{z_\alpha z_\beta} \langle \alpha \beta \rangle, \end{aligned} \tag{4.2}$$

where a is a particle which is not in the collinear set, k is in the collinear set, and $P_{i,j}$ and

$P_{k,l}$ only contain particles from the collinear set. Introducing the following notation⁵

$$\begin{aligned}\Delta(i, j; k) &= \sum_{\alpha=i}^j \sqrt{z_\alpha} \langle k \alpha \rangle, \\ \Delta(i, j; k, l) &= \sum_{\alpha=i}^j \sum_{\beta=k}^l \sqrt{z_\alpha z_\beta} \langle \alpha \beta \rangle,\end{aligned}\tag{4.3}$$

Eqs. (4.2) become

$$\begin{aligned}\langle k P_{i,j} \rangle &\rightarrow [P \eta] \Delta(i, j; k), \\ \langle P_{i,j} P_{k,l} \rangle &\rightarrow [P \eta]^2 \Delta(i, j; k, l).\end{aligned}\tag{4.4}$$

As already stated in section 2, η is an arbitrary spinor that can be thought as parametrising the gauge-dependence of a quantity. As in the collinear limit the only η dependence is in $[P\eta]$, we can neglect these spinor products, i.e. , we set $[P\eta] = 1$. The procedure presented in refs. [31, 32] to extract splitting amplitudes goes as follows:

Rule 1 (Collinear limits in the CSW formalism)

1. Consider all CSW diagrams contributing to the collinear limit of the $(n+3)$ -point amplitude $A_{n+3}(1, \dots, n, (n+1)^+, (n+2)^{-h}, (n+3)^-)$. This set is obtained by including the diagrams where all the scalar propagators go on-shell in the collinear limit, or equivalently, the diagrams where $n+1$, $n+2$ and $n+3$ are attached to the same CSW vertex.
2. Go to the collinear limit by applying the rules in eq. (4.2).
3. Divide by the hard four-point amplitude $A_4(P^h, (n+1)^+, (n+2)^{-h}, (n+3)^-)$.

In the rest of this section we show how it is possible to extend this procedure to the extraction of antenna functions. To do so, first one has to find the CSW diagrams that contribute to a given singular limit, i.e. , the CSW pole structures of the antenna function has to be identified.

Let us start with the soft limits in the CSW construction. To be concrete, let us consider an $(n+4)$ -point gluon amplitude $A_{n+4}(a, 1, \dots, n, b, c, d)$, where the gluons $1, \dots, n$ become soft. The amplitude then factorises according to eq. (3.8). The soft factor can be easily calculated using the following result [11],

Rule 2 (Gluon insertion rule)

In the soft limit, only Feynman diagrams where the soft gluons are radiated from the external legs of the hard amplitude contribute.

The proof of the gluon insertion rule, based on simple power counting arguments, is presented in appendix B.

⁵These notations differ slightly from those used in refs. [31, 32].

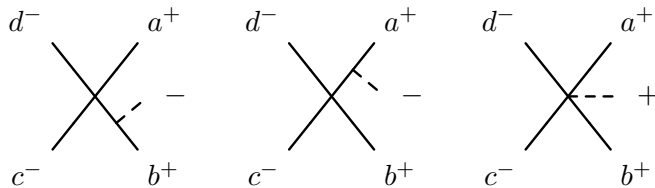


Figure 1: Possible CSW diagrams where a soft gluon is radiated between the external legs a and b . The dashed line corresponds to a soft gluon.

Rule 2 allows an easy identification of the Feynman diagrams corresponding to the emission of additional soft gluons. In order to establish the CSW pole structure of the soft emission, we need to generalize this result to CSW diagrams. Let us consider an $(n + 4)$ -point gluon amplitude $A_{n+4}(a, 1, \dots, n, b, c, d)$, where the gluons $1, \dots, n$ become soft. Rule 2 states that in the soft limit only those Feynman diagrams contribute where the soft particles are radiated from the external legs a and b . However, it is easy to see that, given the helicity configuration, only negative-helicity gluons can be radiated from the external legs in the CSW formalism (a positive-helicity gluon would lead to a three-point CSW vertex with two positive-helicity gluons). The positive-helicity gluons must thus be radiated only from the CSW vertex which builds up the hard amplitude. Note that this is consistent with the gluon insertion rule (cf. the soft factorisation of a pure MHV-amplitude). Similar considerations hold true for different helicity assignments of the particles a, b, c, d in the hard amplitude.

This leads us to the following new formulation of the gluon insertion rule, valid for CSW diagrams (See figure 1),

Rule 3 (Gluon insertion rule in the CSW formalism)

In the soft limit, only CSW diagrams where negative-helicity soft gluons are radiated from the external legs a and b or positive-helicity soft gluons from the CSW vertex that forms the hard amplitude contribute.

Note that the gluon insertion rule is very restrictive on the possible diagrams (figure 1), because it forces the hard gluons c and d to be attached to the same CSW vertex. Thus only a small number of CSW diagrams contributes in the soft limit. Furthermore it is easy to see that, similar to rule 1, this class of diagrams is exactly that where all scalar propagators go on-shell.

We now turn to the CSW pole structure of soft limits. Consider the situation where the set $\{1, \dots, n\}$ contains n_s negative-helicity gluons. The set $\{a, b, c, d\}$ contains two additional negative-helicity gluons, so that the hard amplitude $A_4(a, b, c, d)$ is an MHV-amplitude. The number of CSW propagators in the $(n + 4)$ -point amplitude is then $p = (n_s + 2) - 2 = n_s$. As the hard four-point amplitude does not contain any CSW propagator, we come to the conclusion that a soft factor containing n_s negative-helicity gluons has the following CSW pole structure

$$\text{Soft} \sim \frac{1}{[\]^{n_s}} f(\langle \ \rangle). \quad (4.5)$$

We can extend this result to the case of color-connected soft/collinear limits.⁶ Let us consider to this effect the amplitude $A_{n+3}(a, 1, \dots, k, \dots, n, b, c)$ in the limit where $1, \dots, k$ are soft and $k+1, \dots, n$ are collinear. In this limit the amplitude can be approximated by [9]

$$A_{n+3}(a, 1, \dots, k, \dots, n, b, c) \sim \sum_h \mathcal{S}(a; 1, \dots, k; k+1, \dots, n) \text{Split}_{-h}(k+1, \dots, n) A_4(a, P^h, b, c), \quad (4.6)$$

where \mathcal{S} denotes a universal soft factor. In ref. [9] it was shown that \mathcal{S} can be obtained by taking the leading soft behavior in the splitting amplitude $\text{Split}_{-h}(1, \dots, n)$, i.e.

$$\text{Split}_{-h}(1, \dots, n) \sim \mathcal{S}(a; 1, \dots, k; k+1, \dots, n) \text{Split}_{-h}(k+1, \dots, n). \quad (4.7)$$

We will now determine the CSW pole structure by applying the gluon insertion rule 3 to the splitting amplitude $\text{Split}_{-h}(k+1, \dots, n)$. From rule 1 we know that in the limit where $k+1, \dots, n$ are collinear, only those CSW diagrams of $A_{n-k+3}(a, k+1, \dots, n, b, c)$ contribute where a, b, c are attached to the same CSW vertex. We now apply the rule 3 to attach soft gluons to this set of diagrams. It is easy to see that

- As soft negative-helicity gluons can only be emitted from the external lines, we add a new divergent propagator to the diagram each time we add a soft negative-helicity gluon.
- For the emission of soft positive-helicity gluons, the number of CSW propagators is left unchanged.

Hence, if n_s denotes the number of soft negative-helicity gluons in the set $\{1, \dots, k\}$, we add n_s new divergent propagators to the diagrams, and so

- if $h = +1$,

$$\mathcal{S}(a; 1, \dots, k; k+1, \dots, n) \text{Split}_{-}(k+1, \dots, n) \sim \frac{1}{[\]_{n_s}} \frac{1}{[\]_{n_-}} \sim \frac{1}{[\]_{n_s+n_-}}, \quad (4.8)$$

- if $h = -1$,

$$\mathcal{S}(a; 1, \dots, k; k+1, \dots, n) \text{Split}_{+}(k+1, \dots, n) \sim \frac{1}{[\]_{n_s}} \frac{1}{[\]_{n_- - 1}} \sim \frac{1}{[\]_{(n_s+n_-) - 1}}, \quad (4.9)$$

where n_- is the number of negative-helicity gluons in the collinear set $\{k+1, \dots, n\}$.

We can now turn to the case of antenna functions. By definition, an antenna function contains all possible infrared singularities, both soft and collinear, that an amplitude can have if n particles become soft or collinear. We analyze the CSW pole structures of the antenna functions, using the fact that they reproduce the known CSW pole structures in the various soft and collinear limits.

⁶The color-disconnected case is trivial.

Let us start with $\text{Ant}(\hat{a}^-, \hat{b}^- \leftarrow a, 1, \dots, n, b)$, and consider the CSW pole structure of the different soft and collinear singularities contained in this antenna function. Let n_- denote the number of negative helicities in the set $\{1, \dots, n\}$ and N_- the number of negative helicities in the set $\{a, 1, \dots, n, b\}$. Then we have the following possibilities:

- $k_1, \dots, k_n \parallel k_a$

In this limit we have $k_{\hat{a}} \rightarrow P \equiv k_a + k_1 + \dots + k_n$ and $k_{\hat{b}} \rightarrow k_b$, and

$$\text{Ant}(\hat{a}^-, \hat{b}^- \leftarrow a, 1, \dots, n, b) \rightarrow \text{Split}_-(a, 1, \dots, n) \sim \frac{1}{[\]^{n_a}} \sim \frac{1}{[\]^{N_-}}, \quad (4.10)$$

where n_a is the number of negative helicities in the set $\{a, 1, \dots, n\}$, and $n_a = N_-$. Note that if $k_{\hat{b}} \rightarrow k_b$, then $h_b = -h_{\hat{b}} = +1$.

- $k_1, \dots, k_n \parallel k_b$

In this limit we have $k_{\hat{a}} \rightarrow k_a$ and $k_{\hat{b}} \rightarrow P \equiv k_1 + \dots + k_n + k_b$, and

$$\text{Ant}(\hat{a}^-, \hat{b}^- \leftarrow a, 1, \dots, n, b) \rightarrow \text{Split}_-(1, \dots, n, b) \sim \frac{1}{[\]^{n_b}} \sim \frac{1}{[\]^{N_-}}, \quad (4.11)$$

where n_b is the number of negative helicities in the set $\{1, \dots, n, b\}$, and $n_b = N_-$. Since $k_{\hat{a}} \rightarrow k_a$, then $h_a = -h_{\hat{a}} = +1$.

- $k_1, \dots, k_n \rightarrow 0$

In this limit we have $k_{\hat{a}} \rightarrow k_a$ and $k_{\hat{b}} \rightarrow k_b$, and

$$\text{Ant}(\hat{a}^-, \hat{b}^- \leftarrow a, 1, \dots, n, b) \rightarrow \text{Soft}(a, 1, \dots, n, b) \sim \frac{1}{[\]^{n_-}} \sim \frac{1}{[\]^{N_-}}, \quad (4.12)$$

and $n_- = N_-$ because if $k_{\hat{a}} \rightarrow k_a$, then $h_a = -h_{\hat{a}} = +1$ and if $k_{\hat{b}} \rightarrow k_b$, then $h_b = -h_{\hat{b}} = +1$.

- $k_1, \dots, k_{i-1} \parallel k_a$ and $k_i, \dots, k_j \rightarrow 0$ and $k_{j+1}, \dots, k_n \parallel k_b$.

In this limit we have $k_{\hat{a}} \rightarrow k_a + k_1 + \dots + k_{i-1}$ and $k_{\hat{b}} \rightarrow k_{j+1} + \dots + k_n + k_b$, and

$$\begin{aligned} \text{Ant}(\hat{a}^-, \hat{b}^- \leftarrow a, 1, \dots, n, b) &\rightarrow \text{Split}_-(a, 1, \dots, i-1) \text{Split}_-(j+1, \dots, n, b) \\ &\quad \times \mathcal{S}(a, 1, \dots, i-1; i, \dots, j; j+1, \dots, n, b) \\ &\sim \frac{1}{[\]^{n_a}} \frac{1}{[\]^{n_s}} \frac{1}{[\]^{n_b}} \sim \frac{1}{[\]^{N_-}}, \end{aligned} \quad (4.13)$$

where n_a , n_s and n_b are the numbers of negative-helicity gluons in the sets $\{a, 1, \dots, i-1\}$, $\{i, \dots, j\}$ and $\{j+1, \dots, n, b\}$ respectively, and $n_a + n_s + n_b = N_-$.

We are therefore able to conclude that

$$\text{Ant}(\hat{a}^-, \hat{b}^- \leftarrow a, 1, \dots, n, b) \sim \frac{1}{[\]^{N_-}} f(\langle \ \rangle). \quad (4.14)$$

Similar arguments for the other helicity assignments lead to

$$\begin{aligned}
 \text{Ant}(\hat{a}^-, \hat{b}^+ \leftarrow a, 1, \dots, n, b) &\sim \frac{1}{[\]_{N_- - 1}} f(\langle \ \ \rangle), \\
 \text{Ant}(\hat{a}^+, \hat{b}^- \leftarrow a, 1, \dots, n, b) &\sim \frac{1}{[\]_{N_- - 1}} f(\langle \ \ \rangle), \\
 \text{Ant}(\hat{a}^+, \hat{b}^+ \leftarrow a, 1, \dots, n, b) &\sim \frac{1}{[\]_{N_- - 2}} f(\langle \ \ \rangle).
 \end{aligned}
 \tag{4.15}$$

Eqs. (4.14)–(4.15) are the analogues for antenna functions of eq. (4.1) for splitting amplitudes. We show now that, as it was already the case for splitting amplitudes, the CSW pole structure allows us to identify a priori the set of CSW diagrams that contributes to the singular limit.

Let us start with $\text{Ant}(\hat{a}^-, \hat{b}^- \leftarrow a, 1, \dots, n, b)$, and consider an $(n+4)$ -point amplitude $A_{n+4}(a, 1, \dots, n, b, c^-, d^-)$. In the limit where the particles $1, \dots, n$ become singular, the amplitude exhibits the factorisation property, eq. (3.11),

$$A_{n+4}(a, 1, \dots, n, b, c^-, d^-) \sim \text{Ant}(\hat{a}^-, \hat{b}^- \leftarrow a, 1, \dots, n, b) A_4(\hat{a}^+, \hat{b}^+, c^-, d^-).
 \tag{4.16}$$

If N_- is the number of negative-helicity gluons in the set $\{a, 1, \dots, n, b\}$, then the number of CSW propagators in the $(n+4)$ -point amplitude is $p = (N_- + 2) - 2 = N_-$. From eq. (4.14) we know that the CSW pole structure of this antenna function is

$$\text{Ant}(\hat{a}^-, \hat{b}^- \leftarrow a, 1, \dots, n, b) \sim \frac{1}{[\]_{N_-}} f(\langle \ \ \rangle),
 \tag{4.17}$$

and thus the CSW diagrams contributing to the antenna function are exactly those where all scalar propagators go on-shell. Similar conclusions can be drawn for the other helicity configurations. This brings us to the first important conclusion that rule 1 can be generalized to antenna functions:

The CSW diagrams that contribute to the singular limit of the $(n+4)$ -point amplitude $A_{n+4}(a, 1, \dots, n, b, c^-, d^-)$ are exactly those where all scalar propagators go on-shell.

5. Diagrammatic evaluation of antenna functions in the CSW formalism

In the previous section we showed how we can identify the CSW diagrams that contribute to the singular limit just by examining the pole structure of the CSW diagrams that build up the amplitude. The simple idea is that not all the CSW diagrams that contribute to an amplitude are relevant in the computation of a given singular limit. This result turns out to be very similar to the corresponding result for splitting amplitudes obtained in refs. [31, 32].

For the antenna functions, however, it is possible to go one step further. We note that keeping only those CSW diagrams where all propagators go on-shell is equivalent to requiring that c and d must be attached to the same m -point CSW vertex with $m \geq 4$. Let

us consider now a specific CSW diagram satisfying this condition. Then the CSW vertex with c and d attached gives a contribution

$$\frac{\langle cd \rangle^3}{\langle P_{j,b} c \rangle \langle d P_{a,i} \rangle}, \quad (5.1)$$

with $(i, j) \neq (b, a)$ because otherwise c and d would be attached to a 3-point CSW vertex, and $i = a$ and / or $j = b$ if a and b are attached to the same CSW vertex as c and d . Consider the following Schouten identity, eq. (2.3),

$$\langle \hat{b}\hat{a} \rangle \langle ca \rangle = \langle \hat{b}a \rangle \langle c\hat{a} \rangle + \langle c\hat{b} \rangle \langle a\hat{a} \rangle \Rightarrow \frac{\langle ca \rangle}{\langle c\hat{a} \rangle} = \frac{\langle \hat{b}a \rangle}{\langle \hat{b}\hat{a} \rangle} + \frac{\langle c\hat{b} \rangle \langle a\hat{a} \rangle}{\langle c\hat{a} \rangle \langle \hat{b}\hat{a} \rangle}, \quad (5.2)$$

where \hat{a} and \hat{b} denote the reconstruction functions defined in the appendix A. It is manifest that, if $\langle a\hat{a} \rangle$ goes to zero in the singular limit, then eq. (5.2) drastically simplifies. To show this, we go into the frame where k_a and k_b are aligned along the same direction. As recalled in appendix A, the reconstruction functions can be chosen to be of the form

$$k_{\hat{a}} = A k_a + B k_b + \sum_{j=1}^n k_j r_j, \quad (5.3)$$

where A , B and r_j are functions of invariants in the particle momenta, so they are unaffected by this specific choice of reference frame. As $k_{\hat{a}}$ is by definition lightlike, we can switch to lightcone coordinates. In the specific reference frame we chose, we get

$$\begin{aligned} k_{\hat{a}}^+ &= A k_a^+ + \sum_{j=1}^n k_j^+ r_j, \\ k_{\hat{a}}^- &= B k_b^- + \sum_{j=1}^n k_j^- r_j, \\ k_{\hat{a}\perp} &= \sum_{j=1}^n k_{j\perp} r_j. \end{aligned} \quad (5.4)$$

Furthermore it is easy to see that in this frame we can write

$$k_j \text{ singular} \Leftrightarrow k_{j\perp} \rightarrow 0, \quad \forall j = 1, \dots, n. \quad (5.5)$$

The spinor product $\langle a\hat{a} \rangle$ can now be evaluated in this frame using the lightcone coordinates [12],

$$\langle a\hat{a} \rangle = -i \sqrt{\frac{-k_a^+}{k_{\hat{a}}^+}} k_{\hat{a}\perp} = -i \sqrt{\frac{-k_a^+}{A k_a^+ + \sum_{j=1}^n k_j^+ r_j}} \sum_{j=1}^n k_{j\perp} r_j. \quad (5.6)$$

By definition of the reconstruction functions, $k_{\hat{a}}$ and k_a are collinear in every singular limit, and so the square root gives just a phase in the singular limit. Thus, due to eq. (5.5), $\langle a\hat{a} \rangle$ goes to zero in the singular limit unless there are poles in the coefficients r_j that

could prevent the product $r_j k_{j\perp}$ from going to zero as $k_{j\perp} \rightarrow 0$. Recalling the analytic expression for the coefficients, eq. (A.6), we see that r_j may contain a pole if $k_j \cdot K \rightarrow 0$. As K always contains the momenta of the two hard particles a and b , $k_j \cdot K \rightarrow 0$ if and only if $k_j \rightarrow 0$. However, a quick look at eq. (A.6) shows that r_j does not contain a pole in this limit, so r_j does not contain a pole in any limit. This concludes the proof that $\langle a\hat{a} \rangle$ goes to zero in the singular limit, and allows us to conclude that in the singular limit

$$\frac{\langle ca \rangle}{\langle c\hat{a} \rangle} \rightarrow \frac{\langle \hat{b}a \rangle}{\langle \hat{b}\hat{a} \rangle}. \quad (5.7)$$

Similar conclusions can of course be drawn for $\langle b\hat{b} \rangle$. We can also analyze what happens to the spinor product $\langle P_{j,b} c \rangle$ in the singular limit.

- If $j = b$, then $\langle P_{j,b} c \rangle = \langle bc \rangle$, and in the singular limit

$$\langle bc \rangle \rightarrow \langle \hat{b}c \rangle \frac{\langle \hat{a}b \rangle}{\langle \hat{a}\hat{b} \rangle}. \quad (5.8)$$

- If $j \neq b$, then $\langle P_{j,b} c \rangle = \langle bc \rangle [b\eta] + \sum_{k=j}^n \langle kc \rangle [k\eta]$. The first term has already been dealt with in eq. (5.8). The remaining terms can be rewritten using Schouten-identity

$$\langle kc \rangle = \langle \hat{b}c \rangle \frac{\langle \hat{a}k \rangle}{\langle \hat{a}\hat{b} \rangle} + \langle \hat{a}c \rangle \frac{\langle k\hat{b} \rangle}{\langle \hat{a}\hat{b} \rangle}. \quad (5.9)$$

The second term on the right-hand side, proportional to $\langle k\hat{b} \rangle$, does not contribute in the singular limit. To see this let us first see what happens in the limit where $k_j, \dots, k_\ell \parallel k_a$ and $k_{\ell+1}, \dots, k_n \parallel k_b$, with $j < \ell \leq b$. Then this CSW diagram would have a non-divergent scalar propagator. As we are only looking for those diagrams where all scalar propagators go on-shell, we can neglect this case. So we only need to analyze the situation where $k_j, \dots, k_n \parallel k_b$. Using lightcone coordinates, we find, $\forall j \leq \ell \leq n$,

$$\langle \ell\hat{b} \rangle = k_{\ell\perp} \sqrt{\frac{k_b^+}{k_\ell^+}} - k_{\hat{b}\perp} \sqrt{\frac{k_\ell^+}{k_b^+}}. \quad (5.10)$$

We know already that in the singular limit

$$k_{\hat{b}}^+ \rightarrow 0, \quad k_{\hat{b}\perp} \rightarrow 0, \quad k_{\ell\perp} \rightarrow 0, \quad \forall 1 \leq \ell \leq n, \quad (5.11)$$

and $k_{\hat{b}}$ is collinear to k_b in every singular limit. Let us have a closer look at the first term. From eq. (5.11) it follows that in the singular limit $k_{\ell\perp}, k_\ell^+, k_{\hat{b}}^+ \rightarrow 0$, where we used the fact that in our specific choice of reference frame $k_b^+ = 0$. Furthermore, all external particles must fulfill the on-shell condition $k^+ k^- = |k_\perp|^2$, and so

$$\frac{k_{\ell\perp}}{\sqrt{k_\ell^+}} \sim \sqrt{k_\ell^-}, \quad (5.12)$$

and $k_\ell^- \neq 0$ if $k_j \parallel k_b$. So the first term goes to zero as $\sqrt{k_b^+}$. Similar arguments show that also the second term goes to zero in the singular limit, and so we can conclude that $\langle k\hat{b} \rangle$ vanishes. Putting everything together, we obtain

$$\langle P_{j,b}c \rangle \rightarrow \langle \hat{b}c \rangle \frac{\langle \hat{a}b \rangle}{\langle \hat{a}\hat{b} \rangle} [b\eta] + \sum_{k=j}^n \langle \hat{b}c \rangle \frac{\langle \hat{a}k \rangle}{\langle \hat{a}\hat{b} \rangle} [k\eta] = \langle \hat{b}c \rangle \frac{\langle \hat{a}P_{j,b} \rangle}{\langle \hat{a}\hat{b} \rangle}. \quad (5.13)$$

Both cases can be summarized as

$$\langle P_{j,b}c \rangle \rightarrow \langle \hat{b}c \rangle \frac{\langle \hat{a}P_{j,b} \rangle}{\langle \hat{a}\hat{b} \rangle}. \quad (5.14)$$

Similarly one finds

$$\langle dP_{a,i} \rangle \rightarrow \langle d\hat{a} \rangle \frac{\langle P_{a,i}\hat{b} \rangle}{\langle \hat{a}\hat{b} \rangle}. \quad (5.15)$$

The contribution from eq. (5.1) to the singular limit therefore becomes

$$\frac{\langle cd \rangle^3}{\langle P_{j,b}c \rangle \langle dP_{a,i} \rangle} \rightarrow \frac{\langle cd \rangle^3}{\langle \hat{a}\hat{b} \rangle \langle \hat{b}c \rangle \langle d\hat{a} \rangle} \frac{\langle \hat{a}\hat{b} \rangle^3}{\langle P_{j,b}\hat{a} \rangle \langle \hat{b}P_{a,i} \rangle}. \quad (5.16)$$

The first factor on the right-hand side of eq. (5.16) corresponds to the hard four-point amplitude in eq. (4.16). The second factor has the same functional form as the left-hand side, with c and d replaced by \hat{a} and \hat{b} . This proves the following result: The antenna function $\text{Ant}(\hat{a}^-, \hat{b}^- \leftarrow a, 1, \dots, n, b)$ can be calculated by evaluating all CSW diagrams that contribute to the $(n+4)$ -point amplitude $A_{n+4}(a, 1, \dots, n, b, \hat{a}^-, \hat{b}^-)$ and where \hat{a} and \hat{b} are attached to the same m -point CSW vertex, with $m \geq 4$.

In the rest of this section we show how this result can be generalized to the remaining antenna functions. The proof for $\text{Ant}(\hat{a}^+, \hat{b}^+ \leftarrow a, 1, \dots, n, b)$ is similar to the previous case, and so we do not give it explicitly here.

Let us turn to $\text{Ant}(\hat{a}^+, \hat{b}^- \leftarrow a, 1, \dots, n, b)$ and let us consider the $(n+4)$ -point amplitude $A_{n+4}(a, 1, \dots, n, b, c^-, d^+)$. From eq. (3.11) we know that in the singular limit for the particles $1, \dots, n$, the amplitude exhibits the factorisation property

$$A_{n+4}(a, 1, \dots, n, b, c^-, d^+) \longrightarrow \sum_h \text{Ant}(\hat{a}^h, \hat{b}^{-h} \leftarrow a, 1, \dots, n, b) A_4(\hat{a}^{-h}, \hat{b}^h, c^-, d^+). \quad (5.17)$$

If N_- is the number of negative-helicity gluons in the set $\{a, 1, \dots, n, b\}$, then the number of CSW propagators in the $(n+4)$ -point amplitude is $p = (N_- + 1) - 2 = N_- - 1$. Furthermore, from the considerations in section 4 we know that the CSW pole structure of this antenna function is

$$\text{Ant}(\hat{a}^+, \hat{b}^- \leftarrow a, 1, \dots, n, b) \sim \frac{1}{[\]_{N_- - 1}}, \quad (5.18)$$

and thus the CSW diagrams contributing to the antenna function are exactly those where c and d are attached to the same n -point CSW vertex, with $n \geq 4$. The contribution from the CSW vertex that contains c and d is of the form

$$\frac{\langle cP_{\alpha\beta} \rangle^4}{\langle P_{j,b}c \rangle \langle cd \rangle \langle dP_{a,i} \rangle}, \quad (5.19)$$

where $P_{\alpha\beta}$ denotes the momentum of the second negative helicity leg attached to this vertex, and $\alpha = \beta$ for an external leg. We know already that in the singular limit we have

$$\langle P_{j,b} c \rangle \rightarrow \langle \hat{b}c \rangle \frac{\langle \hat{a}P_{j,b} \rangle}{\langle \hat{a}\hat{b} \rangle}, \quad (5.20)$$

$$\langle d P_{a,i} \rangle \rightarrow \langle d\hat{a} \rangle \frac{\langle P_{a,i} \hat{b} \rangle}{\langle \hat{a}\hat{b} \rangle}. \quad (5.21)$$

Applying the Schouten-identity (5.2), we get for the spinor product in the numerator

$$\begin{aligned} \langle c P_{\alpha\beta} \rangle &= \sum_{k=\alpha}^{\beta} \langle cx \rangle [k\eta] \\ &= \sum_{k=\alpha}^{\beta} \left(\langle c\hat{a} \rangle \frac{\langle k\hat{b} \rangle}{\langle \hat{a}\hat{b} \rangle} + \langle c\hat{b} \rangle \frac{\langle k\hat{a} \rangle}{\langle \hat{b}\hat{a} \rangle} \right) [k\eta] \\ &= \langle c\hat{a} \rangle \frac{\langle P_{\alpha\beta} \hat{b} \rangle}{\langle \hat{a}\hat{b} \rangle} + \langle c\hat{b} \rangle \frac{\langle \hat{a} P_{\alpha\beta} \rangle}{\langle \hat{a}\hat{b} \rangle} \end{aligned} \quad (5.22)$$

Inserting eq. (5.22) into eq. (5.19), we get five terms:

1. a term proportional to $\langle P_{\alpha\beta} \hat{b} \rangle^4$.
2. a term proportional to $\langle P_{\alpha\beta} \hat{a} \rangle^4$.
3. three “mixed” terms of the form $\langle P_{\alpha\beta} \hat{a} \rangle^q \langle P_{\alpha\beta} \hat{b} \rangle^{4-q}$, $q = 1, 2, 3$.

We separately analyze the different limits:

- If $k_{\alpha}, \dots, k_{\beta} \parallel k_a$, then $k_{\hat{a}} \rightarrow P$ and $k_j \rightarrow z_j P$, $\forall \alpha \leq j \leq \beta$ and so $\langle P_{\alpha\beta} \hat{a} \rangle \rightarrow 0$.
- If $k_{\alpha}, \dots, k_{\beta} \parallel k_b$, then $k_{\hat{b}} \rightarrow P$ and $k_j \rightarrow z_j P$, $\forall \alpha \leq j \leq \beta$ and so $\langle P_{\alpha\beta} \hat{b} \rangle \rightarrow 0$.
- If $k_{\alpha}, \dots, k_m \parallel k_a$ and $k_{m'}, \dots, k_{\beta} \parallel k_b$, $\alpha \leq m < m' \leq \beta$, then the propagator $1/P_{\alpha\beta}^2$ is not divergent, and thus diagrams with a propagator $1/P_{\alpha\beta}^2$ are not divergent enough to contribute to this limit.
- If $k_{\alpha}, \dots, k_{\beta} \rightarrow 0$, the situation is more subtle. We show in appendix D that in this limit only those diagrams contribute where $\alpha = a$, and $P_{\alpha\beta} \rightarrow k_a$. Thus in this limit $\langle P_{\alpha\beta} \hat{a} \rangle \sim \langle a\hat{a} \rangle \rightarrow 0$.

Finally, we see that in any situation either $\langle P_{\alpha\beta} \hat{a} \rangle$ or $\langle P_{\alpha\beta} \hat{b} \rangle$ go to zero, and thus we can drop the “mixed” terms. So in the singular limit eq. (5.19) becomes

$$\begin{aligned} \frac{\langle P_{\alpha\beta} c \rangle^4}{\langle P_{j,b} c \rangle \langle cd \rangle \langle d P_{a,i} \rangle} &\rightarrow \frac{\langle c\hat{a} \rangle^4}{\langle \hat{a}\hat{b} \rangle \langle \hat{b}c \rangle \langle cd \rangle \langle d\hat{a} \rangle} \frac{\langle P_{\alpha\beta} \hat{b} \rangle^4}{\langle P_{j,b} \hat{a} \rangle \langle \hat{a}\hat{b} \rangle \langle \hat{b} P_{a,i} \rangle} \\ &+ \frac{\langle c\hat{b} \rangle^4}{\langle \hat{a}\hat{b} \rangle \langle \hat{b}c \rangle \langle cd \rangle \langle d\hat{a} \rangle} \frac{\langle P_{\alpha\beta} \hat{a} \rangle^4}{\langle P_{j,b} \hat{a} \rangle \langle \hat{a}\hat{b} \rangle \langle \hat{b} P_{a,i} \rangle}, \end{aligned} \quad (5.23)$$

$$\begin{aligned} &\rightarrow A_4(\hat{a}^-, \hat{b}^+, c^-, d^+) \frac{\langle P_{\alpha\beta} \hat{b} \rangle^4}{\langle P_{j,b} \hat{a} \rangle \langle \hat{a} \hat{b} \rangle \langle \hat{b} P_{a,i} \rangle} \\ &+ A_4(\hat{a}^+, \hat{b}^-, c^-, d^+) \frac{\langle P_{\alpha\beta} \hat{a} \rangle^4}{\langle P_{j,b} \hat{a} \rangle \langle \hat{a} \hat{b} \rangle \langle \hat{b} P_{a,i} \rangle}. \end{aligned}$$

We see that the first term contributes to $\text{Ant}(\hat{a}^+, \hat{b}^- \leftarrow a, 1, \dots, n, b)$ and the second term contributes to $\text{Ant}(\hat{a}^-, \hat{b}^+ \leftarrow a, 1, \dots, n, b)$. Note that again the second factor in each term has the same functional form as the left-hand side, so we proved our claim that $\text{Ant}(\hat{a}^{h_a}, \hat{b}^{h_b} \leftarrow a, 1, \dots, n, b)$ can be calculated by evaluating all CSW diagrams that contribute to the $(n+4)$ -point amplitude $A_{n+4}(a, 1, \dots, n, b, \hat{a}^{h_a}, \hat{b}^{h_b})$ and where \hat{a} and \hat{b} are attached to the same m -point CSW vertex, with $m \geq 4$.

6. Discussion and results

In the previous section we proved the main result of this paper which states that in the CSW formalism it is possible to define in a natural way antenna functions as a sum of MHV diagrams. This can be summarized as follows:

Rule 4 (Antenna functions in the CSW formalism)

The antenna function $\text{Ant}(\hat{a}^{h_a}, \hat{b}^{h_b} \leftarrow a, 1, \dots, n, b)$ can be calculated by evaluating all CSW diagrams that contribute to the $(n+4)$ -point amplitude $A_{n+4}(a, 1, \dots, n, b, \hat{a}^{h_a}, \hat{b}^{h_b})$ and where \hat{a} and \hat{b} are attached to the same m -point CSW vertex, with $m \geq 4$.

This result allows to directly build all antenna functions using the CSW formalism and the related CSW diagrams. It is possible to identify a priori the CSW diagrams that contribute to the singular limit, and thus it is not necessary to evaluate all the CSW diagrams that contribute to the full amplitude $A_{n+4}(a, 1, \dots, n, b, c, d)$. This class of CSW diagrams is uniquely defined by the requirement that all the CSW propagators must go on-shell in the singular limit, which turns out to be very similar to rule 1 for splitting amplitudes. The advantage of our result is, however, that the functional form of the antenna functions is exactly given by the CSW diagram, whereas in rule 1 one still has to go to the collinear limit. Antenna functions can be represented as a special class of CSW diagrams, the class of diagrams being the one defined by rule 4. Let us make a few comments about the antenna functions obtained from rule 4:

1. The antenna functions may still contain non-divergent pieces, coming from CSW diagrams that are not divergent in any limit. For example, let us consider the antenna function $\text{Ant}(\hat{a}^-, \hat{b}^+ \leftarrow a^-, 1^-, 2^+, b^+)$. In ref. [20] it was shown that this antenna function is zero. Using rule 4 we generate a set of four non zero CSW diagrams. It is easy to check however that all four diagrams are finite in all the singular limits we are interested in.
2. As the antenna functions are built from CSW diagrams, it is easy to see that the antenna functions obtained from rule 4 will always fulfill the reflection identity

$$\text{Ant}(\hat{a}^{h_a}, \hat{b}^{h_b} \leftarrow a, 1, \dots, n, b) = (-1)^n \text{Ant}(\hat{b}^{h_b}, \hat{a}^{h_a} \leftarrow b, n, \dots, 1, a). \quad (6.1)$$

3. The antenna functions are in general dependent of an arbitrary spinor η . In practise, however, η cannot be chosen among the momenta appearing inside the antenna functions, because this would be inconsistent with the assumption that there are no infrared poles in the amplitude coming from antiholomorphic spinor products of the form $[\cdot\eta]$, e.g. if we chose $\eta = a$, then $[1\eta] = [1a]$ could lead to a pole in the limit where these two particles become collinear, which is not included in the class of diagrams defined by rule 4.
4. The proof of our result crucially relies on the fact that $\langle a\hat{a} \rangle$ goes to zero in the singular limit. We were able to show this explicitly for the reconstruction functions given in Eqs (A.1)–(A.2). However, if different reconstruction functions are chosen, it should be checked that this statement still holds true for the new analytic expressions.

In ref. [33] a recursive formulation of the CSW formalism in terms of single and double-line currents was introduced. A review of this recursive algorithm is given in appendix C. As rule 4 allows us to identify a priori the CSW diagrams that contribute to a given singular limit, we can use this result to write down a recursive algorithm for antenna functions in terms of the single and double-line currents. As the antenna functions contain all kinds of collinear and soft singularities, it is straightforward to derive the corresponding recursions for splitting amplitudes and soft factors. This algorithm is discussed in section 8.

General formulas for NMHV antenna functions

In this section we apply rule 4 to derive generic formulas for all MHV and next-to-MHV (NMHV) type antenna functions. We give explicitly the results for antenna functions of the form $\text{Ant}(\hat{a}^-, \hat{b}^- \leftarrow a, 1, \dots, n, b)$ and $\text{Ant}(\hat{a}^-, \hat{b}^+ \leftarrow a, 1, \dots, n, b)$. The remaining antenna functions are related to the previous ones by parity and reflection identity (6.1).

MHV-type antenna functions. The simplest antenna functions are those obtained from a single CSW vertex. Applying rule 4 we immediately find

$$\begin{aligned} \text{Ant}(\hat{a}^-, \hat{b}^- \leftarrow a^+, 1^+, \dots, n^+, b^+) &= A_{n+4}(a^+, 1^+, \dots, n^+, b^+, \hat{a}^-, \hat{b}^-) \\ &= \frac{\langle \hat{a}\hat{b} \rangle^3}{\langle a1 \rangle \langle 12 \rangle \dots \langle nb \rangle \langle b\hat{a} \rangle \langle \hat{b}a \rangle}, \end{aligned} \tag{6.2}$$

$$\begin{aligned} \text{Ant}(\hat{a}^-, \hat{b}^+ \leftarrow a^+, 1^+, \dots, j^-, \dots, n^+, b^+) &= A_{n+4}(a^+, 1^+, \dots, j^-, \dots, n^+, b^+, \hat{a}^-, \hat{b}^+) \\ &= \frac{\langle \hat{a}j \rangle^4}{\langle a1 \rangle \langle 12 \rangle \dots \langle nb \rangle \langle b\hat{a} \rangle \langle \hat{a}\hat{b} \rangle \langle \hat{b}a \rangle}. \end{aligned} \tag{6.3}$$

NMHV-type antenna functions. The NMHV-type antenna functions are derived from CSW diagrams containing exactly one propagator. The CSW diagrams obtained from rule 4 are shown in figure 2 and figure 3. The generic formulas for NMHV-type antenna functions are

$$\begin{aligned} \text{Ant}(\hat{a}^-, \hat{b}^- \leftarrow a^+, 1^+, \dots, j^-, \dots, n^+, b^+) &= \\ \sum_{l=j}^n \frac{\langle \hat{a}\hat{b} \rangle^4 \langle jP_{a,l} \rangle^4}{s_{a,l} \langle P_{a,l}(l+1) \rangle \langle (l+1)\hat{b} \rangle \langle \hat{b}P_{a,l} \rangle \langle P_{a,l}a \rangle \langle \langle al \rangle \rangle \langle lP_{a,l} \rangle} \end{aligned} \tag{6.4}$$

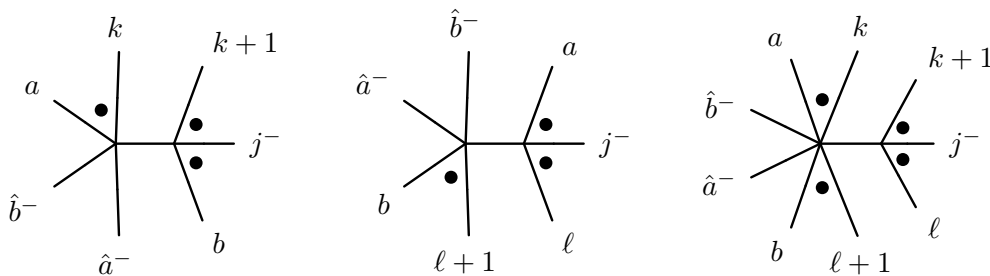


Figure 2: CSW diagrams contributing to $\text{Ant}(\hat{a}^-, \hat{b}^- \leftarrow a^+, 1^+, \dots, j^-, \dots, n^+, b^+)$.

$$\begin{aligned}
 & + \sum_{k=a}^{j-1} \frac{\langle \hat{a}\hat{b} \rangle^4 \langle jP_{k+1,b} \rangle^4}{s_{k+1,b} \langle P_{k+1,b}\hat{a} \rangle \langle \langle \hat{a}k \rangle \rangle \langle kP_{k+1,b} \rangle \langle P_{k+1,b}(k+1) \rangle \langle \langle (k+1)b \rangle \rangle \langle bP_{k+1,b} \rangle} \\
 & + \sum_{k=a}^{j-1} \sum_{l=j}^n \frac{\langle \hat{a}\hat{b} \rangle^4 \langle jP_{k+1,l} \rangle^4}{s_{k+1,l} \langle P_{k+1,l}(l+1) \rangle \langle \langle (l+1)k \rangle \rangle \langle kP_{k+1,l} \rangle \langle P_{k+1,l}(k+1) \rangle \langle \langle (k+1)l \rangle \rangle \langle lP_{k+1,l} \rangle},
 \end{aligned}$$

$$\text{Ant}(\hat{a}^-, \hat{b}^+ \leftarrow a^+, 1^+, \dots, i^-, \dots, j^-, \dots, n^+, b^+) = \tag{6.5}$$

$$\begin{aligned}
 & \sum_{k=a}^{i-1} \frac{\langle \hat{a}P_{k+1,b} \rangle^4 \langle ij \rangle^4}{s_{k+1,b} \langle P_{k+1,b}\hat{a} \rangle \langle \langle \hat{a}k \rangle \rangle \langle kP_{k+1,b} \rangle \langle P_{k+1,b}(k+1) \rangle \langle \langle (k+1)b \rangle \rangle \langle bP_{k+1,b} \rangle} \\
 & + \sum_{l=j}^n \frac{\langle \hat{a}P_{a,l} \rangle^4 \langle ij \rangle^4}{s_{a,l} \langle P_{a,l}(l+1) \rangle \langle \langle (l+1)\hat{b} \rangle \rangle \langle \hat{b}P_{a,l} \rangle \langle P_{a,l}a \rangle \langle \langle al \rangle \rangle \langle lP_{a,l} \rangle} \\
 & + \sum_{k=a}^{i-1} \sum_{l=j}^n \frac{\langle \hat{a}P_{k+1,l} \rangle^4 \langle ij \rangle^4}{s_{k+1,l} \langle P_{k+1,l}(l+1) \rangle \langle \langle (l+1)k \rangle \rangle \langle kP_{k+1,l} \rangle \langle P_{k+1,l}(k+1) \rangle \langle \langle (k+1)l \rangle \rangle \langle lP_{k+1,l} \rangle} \\
 & + \sum_{k=i}^{j-1} \sum_{l=j}^b \frac{\langle \hat{a}i \rangle^4 \langle jP_{k+1,l} \rangle^4}{s_{k+1,l} \langle P_{k+1,l}(l+1) \rangle \langle \langle (l+1)k \rangle \rangle \langle kP_{k+1,l} \rangle \langle P_{k+1,l}(k+1) \rangle \langle \langle (k+1)l \rangle \rangle \langle lP_{k+1,l} \rangle} \\
 & + \sum_{k=\hat{a}}^{i-1} \sum_{l=i}^{j-1} \frac{\langle \hat{a}j \rangle^4 \langle iP_{k+1,l} \rangle^4}{s_{k+1,l} \langle P_{k+1,l}(l+1) \rangle \langle \langle (l+1)k \rangle \rangle \langle kP_{k+1,l} \rangle \langle P_{k+1,l}(k+1) \rangle \langle \langle (k+1)l \rangle \rangle \langle lP_{k+1,l} \rangle},
 \end{aligned}$$

where we introduced the notations

$$\begin{aligned}
 s_{i,j} &= (p_i + p_{i+1} + \dots + p_j)^2, \\
 \langle \langle ij \rangle \rangle &= \prod_{k=i}^{j-1} \langle k(k+1) \rangle.
 \end{aligned} \tag{6.6}$$

Note that the above generic formulas for NMHV-type antenna functions (together with the results for the MHV-type antenna functions) are sufficient to derive the full set of NNLO and N³LO gluon antenna functions, listed in appendix E. We checked explicitly that these antenna functions reproduce the correct infrared limits and that the limits are numerically independent of η using the `Mathematica` package `S@M` [44]. Note that our antenna functions have a slightly simpler and more compact analytic form than those presented in ref. [20]. This fact might simplify the phase space integration of the counterterms when antenna functions are used in a subtraction method.

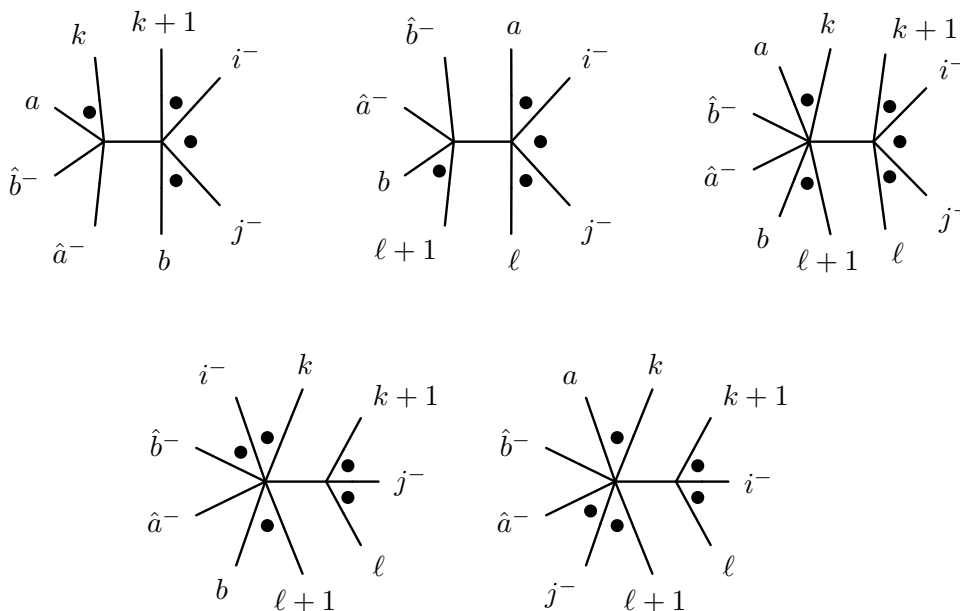


Figure 3: CSW diagrams contributing to $\text{Ant}(\hat{a}^-, \hat{b}^+ \leftarrow a^+, 1^+, \dots, i^-, \dots, j^-, \dots, n^+, b^+)$.

7. From antenna functions to splitting amplitudes

In this section we show how the antenna functions obtained from rule 4 can be used to derive rule 1 for splitting amplitudes presented in the previous section. Let us start with $\text{Split}_-(a, 1, \dots, n)$. We know that for the antenna function $\text{Ant}(\hat{a}^-, \hat{b}^- \leftarrow a, 1, \dots, n, b)$, we have in the limit where $1, \dots, n$ become collinear to a ,

$$\text{Ant}(\hat{a}^-, \hat{b}^- \leftarrow a, 1, \dots, n, b) \longrightarrow \text{Split}_-(a, 1, \dots, n). \tag{7.1}$$

Furthermore, we know that the CSW pole structure of the antenna function built from rule 4 is

$$\text{Ant}(\hat{a}^-, \hat{b}^- \leftarrow a, 1, \dots, n, b) \sim \frac{1}{[\]_{N_-}} f(\langle \ \rangle). \tag{7.2}$$

In the limit under consideration, we have $N_- = n_a$ because in this limit $k_{\hat{b}} \rightarrow k_b$, and so $h_b = +1$. Thus only those CSW diagrams in $\text{Ant}(\hat{a}^-, \hat{b}^- \leftarrow a, 1, \dots, n, b)$ contribute where all $N_- = n_a$ CSW propagators go on-shell in the collinear limit. These diagrams correspond exactly to those where b is attached to the same CSW vertex as \hat{a} and \hat{b} , which is equivalent to rule 1. The derivation of the corresponding result for Split_+ is similar to the Split_- case, so we do not report its derivation.

Note that unlike antenna functions, the splitting amplitudes are uniquely defined. The arbitrariness in the definition of the antenna function is lost when a specific collinear limit is taken, because the class of CSW diagrams contributing to the antenna function can be divided into two different subclasses:

1. The CSW diagrams where b is attached to the same CSW vertex as \hat{a} and \hat{b} : These diagrams are divergent in the collinear limit.⁷

⁷Note that these diagrams may still contain subleading pieces.

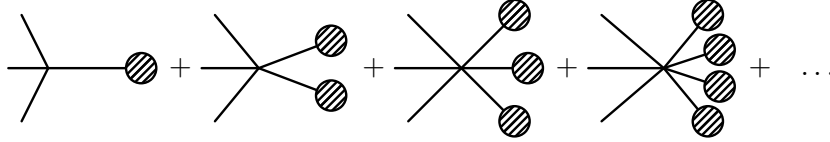


Figure 4: Diagrammatic expansion of the splitting amplitude. The blobs indicate CSW diagrams with a smaller number of legs.

2. The CSW diagrams where b is not attached to the same CSW vertex as \hat{a} and \hat{b} : These diagrams are not divergent enough and are omitted. They contribute to the arbitrary piece from the antenna function.

Diagrammatic approach to splitting amplitudes

In the previous section we showed that in the limit where $1, \dots, n$ become collinear to a only those CSW diagrams of the antenna function contribute where \hat{a} , \hat{b} and b are attached to the same CSW vertex, which is equivalent to the result derived in ref. [31, 32]. In this section we show how this result can be interpreted in terms of CSW diagrams, where the vertices are modified.

The set of diagrams defined by this condition can be easily expanded in terms of the structure of the collinear poles appearing in the diagram (i.e. , $1/s_{a,n}$, $1/s_{a,k}s_{k+1,n}$, etc.). This expansion is shown in figure 4. We find

$$\text{Split}_{-h}(a, 1, \dots, n)A_4(P^h, b, c, d) \sim \sum_{\text{partitions}} \sum_j V_{\pi,j} \prod_{\pi_i} \frac{1}{s_{\pi_i}} D_{j,(\pi_i)}^{\text{MHV}}, \quad (7.3)$$

where the first sum goes over all partitions (including the set $\{a, 1, \dots, n\}$ itself) $\pi = (\pi_i)$ of the set $\{a, 1, \dots, n\}$, and the second sum runs over all diagrams corresponding to this partition. $V_{\pi,j}$ is the MHV vertex which \hat{a} , \hat{b} and b are attached to and s_{π_i} denotes the invariant formed out of the momenta which are in the subset π_i . Note that in this way we reproduce the pole structure of the splitting amplitude: For the partition into one subset, $\pi = \{a, 1, \dots, n\}$, the pole is $1/s_{a,n}$, for the partition into two subsets, $\pi = \{(a, 1, \dots, k), (k+1, \dots, n)\}$, the pole is $1/s_{a,k}s_{k+1,n}$, and so on.

The diagrams $D_{j,(\pi_i)}^{\text{MHV}}$ are CSW diagrams. They however still contain pieces that are subleading in the collinear limit. We show how it is possible to modify the definition of the CSW vertices such that the diagrams in eq. (7.3) only contain the leading collinear pole. Let us consider a specific diagram $D_{j,(\pi_i)}^{\text{MHV}}$. All the vertices in $D_{j,(\pi_i)}^{\text{MHV}}$ only depend on

- particles from the collinear set $a, 1, 2, \dots, n$.
- off-shell legs of the form $P_{k,\ell}$, where k and ℓ are in the collinear set.

Let us first consider the case of a vertex which only depends on off-shell continued legs. We will give as an example the four-point vertex. The generalization is straightforward. An example of a four-point MHV-vertex with all legs continued off-shell is

$$A_4(I^-, J^-, K^+, L^+) = \frac{\langle IJ \rangle^4}{\langle IJ \rangle \langle JK \rangle \langle KL \rangle \langle LI \rangle}, \quad (7.4)$$

where we used the multiindex notation introduced in appendix C, i.e. $A_4(I^-, J^-, K^+, L^+)$ has to be understood as $A_4(P_I^-, P_J^-, P_K^+, P_L^+)$. Applying eq. (4.4), this vertex gives the following contribution in the collinear limit

$$A_4(I^-, J^-, K^+, L^+) \rightarrow \frac{\Delta(I, J)^4}{\Delta(I, J)\Delta(J, K)\Delta(K, L)\Delta(L, I)}, \quad (7.5)$$

The Δ -function has been defined in eq. (4.3), with

$$\Delta(I, J) \equiv \Delta(i_1, i_2; j_1, j_2). \quad (7.6)$$

We would now like to define the object on the right-hand side of eq. (7.5) as an effective MHV vertex in the collinear limit, from which splitting amplitudes could be built. The idea rests on the following observations:

1. An on-shell particle with momentum p_i can be seen as labelled by the multiindex $\bar{i} = (i, i)$, defined in eq. (C.10).
2. The Δ -functions are not independent, but it is easy to see that

$$\begin{aligned} \Delta(\bar{i}, k) &= \sqrt{z_i} \langle ki \rangle, \\ \Delta(\bar{i}, J) &= \sqrt{z_i} \Delta(J, i) = -\Delta(J, \bar{i}), \\ \Delta(\bar{i}, \bar{j}) &= \sqrt{z_i} \Delta(\bar{j}, i) = \sqrt{z_i z_j} \langle ij \rangle. \end{aligned} \quad (7.7)$$

Thus we would like to extend the definition eq. (7.5) to the case where on-shell particles are present in the vertex. For example, if the first particle (with negative helicity) in A_4 is on-shell, $I = \bar{i} = (i, i)$, then, applying eq. (7.7),

$$\begin{aligned} A_4(i^-, J^-, K^+, L^+) &= A_4(\bar{i}^-, J^-, K^+, L^+) \\ &\rightarrow \frac{\Delta(\bar{i}, J)^4}{\Delta(\bar{i}, J)\Delta(J, K)\Delta(K, L)\Delta(L, \bar{i})} \\ &\rightarrow z_i \frac{-\Delta(J, i)^4}{\Delta(J, i)\Delta(J, K)\Delta(K, L)\Delta(L, i)}. \end{aligned} \quad (7.8)$$

On the other hand, a direct application of the collinear rules (4.2) leads to

$$A_4(i^-, J^-, K^+, L^+) \rightarrow \frac{-\Delta(J, i)^4}{\Delta(J, i)\Delta(J, K)\Delta(K, L)\Delta(L, i)}. \quad (7.9)$$

The difference between eqs. (7.8) and (7.9) amounts to a factor z_i . This can be reabsorbed into the one-point current attached to the vertex, by defining the “wave function”

$$J_-(i) = \frac{1}{z_i}, \quad (7.10)$$

which cancels the factor z_i in eq. (7.8). It is easy to see that for an on-shell particle with positive helicity the wave function must then be defined as

$$J_+(i) = z_i. \quad (7.11)$$

and finally that eq. (7.10) and (7.11) can be summarised as

$$J_h(i) = z_i^h. \tag{7.12}$$

The diagram $D_{j,(\pi_i)}^{\text{MHV}}$ has however an additional external leg, corresponding to the incoming momentum P_{π_i} . We define the wave function of this external leg as

$$J_h(P_{\pi_i}) = 1. \tag{7.13}$$

We now turn to the vertex $V_{\pi,j}$ in eq. (7.3). Two cases are to be identified, corresponding to the value of h in eq. (7.3).

1. If $h = -1$, then

$$V_{\pi,j} = \frac{\langle \hat{a} \hat{b} \rangle^3}{\langle \hat{b} P_{\pi_1} \rangle \langle P_{\pi_1} P_{\pi_k} \rangle \langle P_{\pi_k} b \rangle \langle b \hat{a} \rangle}, \tag{7.14}$$

where k denotes the length of the partition. Using the properties of the reconstruction functions and the definition of the Δ -function, we find

$$V_{\pi,j} \rightarrow \frac{1}{z_{\pi_1} z_{\pi_k}} \prod_{\ell=1}^{k-1} \frac{1}{\Delta(P_{\pi_\ell}, P_{\pi_{\ell+1}})}, \tag{7.15}$$

where we defined

$$z_{\pi_j} \equiv \sum_{\ell \in \pi_j} z_\ell. \tag{7.16}$$

2. If $h = +1$, then

$$V_{\pi,j} = \frac{\langle M \hat{b} \rangle^3}{\langle \hat{b} P_{\pi_1} \rangle \langle P_{\pi_1} P_{\pi_k} \rangle \langle P_{\pi_k} b \rangle \langle b \hat{a} \rangle}, \tag{7.17}$$

where now M denotes the propagator with negative helicity attached to $V_{\pi,j}$. This yields

$$V_{\pi,j} \rightarrow \frac{z_M}{z_{\pi_1} z_{\pi_k}} \prod_{\ell=1}^{k-1} \frac{1}{\Delta(P_{\pi_\ell}, P_{\pi_{\ell+1}})}, \tag{7.18}$$

Putting everything together, we can write down the following diagrammatic formula for splitting amplitudes:

$$\text{Split}_{-h}(a, 1, \dots, n) = \sum_{\text{partitions}} \sum_j \mathcal{V}_{\pi,j}^{(h)} \prod_{\pi_i} \frac{1}{s_{\pi_i}} D_j^{(\pi_i)}, \tag{7.19}$$

where

$$\mathcal{V}_{\pi,j}^{(h)} = \frac{z_M^{(1+h)/2}}{z_{\pi_1} z_{\pi_k}} \prod_{\ell=1}^{k-1} \frac{1}{\Delta(P_{\pi_\ell}, P_{\pi_{\ell+1}})}. \tag{7.20}$$

8. Recursive relations

Recursive relations for antenna functions

In this section we apply the recursive formulation of the CSW formalism introduced in ref. [33] and reviewed in appendix C to the calculation of the antenna function. From rule 4, we build the antenna function $\text{Ant}(\hat{a}^{h_{\hat{a}}}, \hat{b}^{h_{\hat{b}}} \leftarrow a, 1, \dots, n, b)$ by calculating all CSW diagrams that contribute to $A_{n+4}(a, 1, \dots, n, b, \hat{a}^{h_{\hat{a}}}, \hat{b}^{h_{\hat{b}}})$ and where \hat{a} and \hat{b} are attached to the same n -point CSW vertex, $n \geq 4$. In the language of the recursive formulation of the CSW formalism this amplitude can be built recursively using eq. (C.14)–(C.15), and by putting the off-shell leg on-shell. This recursion relies on the fact that an arbitrary n -point MHV amplitude can be built recursively out of a small set of building blocks, and we introduce a diagrammatic representation for this construction in terms of double-lines (See figure 8). Hence, the diagrammatic representation of the recursion contains two types of lines, single-lines representing ordinary CSW propagators as well as double-lines arising from the recursive construction of the MHV vertices.

The part of this single-line current where \hat{a} and \hat{b} are attached to the same n -point CSW vertex, $n \geq 4$, corresponds exactly to those diagrams where \hat{a} and \hat{b} are attached to the same double-line current (See figure 5). We can then write immediately

$$\text{Ant}(\hat{a}^{h_{\hat{a}}}, \hat{b}^{h_{\hat{b}}} \leftarrow a, 1, \dots, n, b) = \sum_{\substack{U < V \\ v_2=b}} \sum_M V_4(U, V, \hat{a}, \hat{b}; M_1, M_2) \epsilon^{h_{\hat{a}} h_{\hat{b}}} J_{UV}^{(2+h_{\hat{a}}+h_{\hat{b}})/2}(a, 1, \dots, n, b) \quad (8.1)$$

where

$$\epsilon^{h_{\hat{a}} h_{\hat{b}}} = \begin{cases} \delta_{M_1}^{\hat{a}} \delta_{M_2}^{\hat{b}}, & \text{if } h_{\hat{a}} = h_{\hat{b}} = -1, \\ \delta_{M_2}^{\hat{a}}, & \text{if } h_{\hat{a}} = -h_{\hat{b}} = -1, \\ \delta_{M_2}^{\hat{b}}, & \text{if } h_{\hat{a}} = -h_{\hat{b}} = +1, \\ 1 & \text{if } h_{\hat{a}} = h_{\hat{b}} = +1. \end{cases} \quad (8.2)$$

Note that this recursive relation follows the same spirit as that proposed by Kosower in ref. [20],

1. Build all the single and double-line currents using the recursive relations eqs. (C.14)–(C.15).
2. Calculate the antenna function by using eq. (8.1).

Note also that we could have used the recursive relations in a different way to calculate the antenna function:

1. First build the full amplitude $A_{n+4}(a, 1, \dots, n, b, c, d)$.
2. Second extract the infrared divergent piece in the limit were $1, \dots, n$ are unresolved, i.e. , the antenna function $\text{Ant}(\hat{a}, \hat{b} \leftarrow a, 1, \dots, n, b)$.

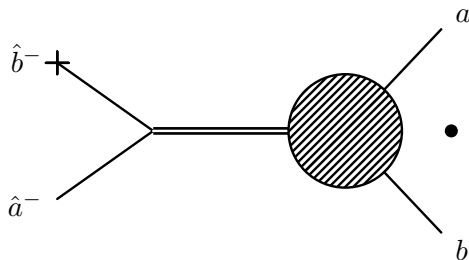


Figure 5: Contribution of $J_+(a, 1, \dots, n, b, c^-)$ in the singular limit. The off-shell leg is denoted by a cross, and the double-line represents the CSW vertex which \hat{a} and \hat{b} are attached to.

The calculation of the full amplitude $A_{n+4}(a, 1, \dots, n, b, c, d)$ needs the evaluation of the $(n+3)$ -point single-line current $J(a, 1, \dots, n, b, c)$, which contains $(n+2)$ -point double-line currents as subcurrents. Eq. (8.1) proves that we only need to evaluate the $(n+2)$ -point double-line current $J_{UV}(a, 1, \dots, n, b)$. This current contains all the information that is needed to build the antenna functions, i.e. , we do not need to solve the full recursion, but we stop after we have built the infrared divergent piece.

Having a recursive formula for antenna functions we can go on and derive recursive relations for splitting amplitudes, by taking the corresponding limits on eq. (8.1). The rest of this section will be devoted to this task.

Recursive relations for splitting amplitudes

In this section we derive the recursive relations for splitting amplitudes by taking the collinear limit of the corresponding recursive relations for the antenna functions, eq. (8.1). We already know that in the limit where $a, 1, \dots, n$ are collinear, only those CSW diagrams contribute where \hat{a}, \hat{b} and b are attached to the same CSW vertex. In terms of the single and double-line currents, this means that \hat{a}, \hat{b} and b must be attached to the same double-line current (See figure 6). In addition we can recast the splitting function as a sum over products of CSW diagrams, where the CSW vertices are replaced by Δ functions and the external particles have “wave functions” corresponding to the momentum fraction z_i .

Combining these two observations with the recursive relations for the antenna functions introduced in the previous section, we are naturally led to a set of recursive relations for the splitting amplitudes themselves. We find

$$\text{Split}_{-h}(a, 1, \dots, n) = J_h(a, 1, \dots, n) + \sum_{\substack{U < V, M \\ v_2 = n}} V_c^{-h}(U, V) J_{UV}^{(1-h)/2}(a, 1, \dots, n). \quad (8.3)$$

Let us conclude this section with a few comments.

1. The recursive relations state that a splitting amplitude can be written as a sum of a single and a double-line current. These currents fulfill formally the same recursive

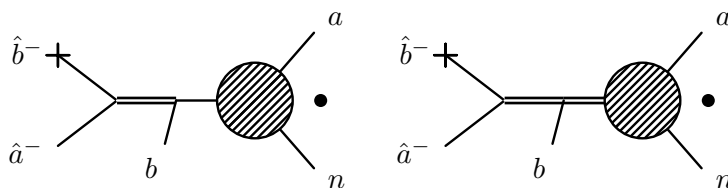


Figure 6: The contribution from $\text{Ant}(\hat{a}^-, \hat{b}^- \leftarrow a, 1, \dots, n, b)$ in the limit $k_1, \dots, k_n \parallel k_a$. The off-shell leg is denoted by a cross, and the double-line represents the CSW vertex to which \hat{a} and \hat{b} are attached.

relations as gluon amplitudes and antenna functions, the vertices however correspond to the CSW vertices in the collinear limit introduced in eq. (7.5) and the one-point currents correspond to the “wave functions” introduced in section 7,

$$J_h(i) = z_i^h. \tag{8.4}$$

2. Once again we could use the recursive relations, eqs. (C.14)–(C.15), in a different way to calculate an n -point splitting amplitude:
 - (a) First calculate the full $n + 3$ point amplitude $A_{n+3}(1, \dots, n, a, b, c)$.
 - (b) Then extract the splitting amplitudes according to

$$A_{n+3}(1, \dots, n, a, b, c) \sim \text{Split}_{-h}(1, \dots, n) A_4(P^h, a, b, c). \tag{8.5}$$

The calculation of the full amplitude needs the calculation of an $(n + 2)$ -point single-line current $J(1, \dots, n, a, b)$, which contains the $(n + 1)$ -point double-line current as a subcurrent. On the other hand, eq. (8.3) tells us that it is sufficient to evaluate n -point single and double-line currents to obtain an n -point splitting amplitude.

3. We checked the splitting amplitudes calculated using these recursive relations against the pure gluon splitting amplitudes obtained in refs. [12, 31]. For all of them we found complete agreement.

9. Conclusion

In this paper we have applied the techniques inspired by the twistor approach to the computation of antenna functions that describe the complete infrared behaviour of multi-gluon amplitudes. Our findings can be summarised as follows.

Our first important result is that antenna functions can be directly written as a sum of a well-defined and limited subset of MHV diagrams among those contributing to a full QCD amplitude. This method leads to very compact expressions for the antenna functions, which might be more suitable for analytic integration over the singular phase space regions, needed in higher order computations. It also gives a very natural way of parametrizing, by

means of an antiholomorphic spinor η , the intrinsic arbitrariness of the antenna function outside the singular regions. As an application, we have shown that it is straightforward to provide explicit expressions for MHV and NMHV antenna functions at all orders, which are sufficient to build the full set of gluon antenna functions up to $N^3\text{LO}$. Although the knowledge of these multi-leg antenna functions might not play a practical role at present for QCD calculation beyond leading order, their knowledge could be of interest, for instance, in testing the infrared structure of recently introduced conjectures for gluon amplitudes at all order in MSYM [45].

The special set of MHV rules relevant for building antenna functions can be easily recast in a recursive form, following the same approach already introduced in ref. [33]. This recursion is formulated in terms of currents and has a similar structure as the Berends-Giele recursive relations. Symbolically, the recursion can be written as

$$\begin{aligned}
 J_1(n) &= \text{---} \text{Y} + \text{---} \text{Y} \text{=} , \\
 J_2(n) &= \text{=} \text{Y} + \text{=} \text{Y} \text{=} ,
 \end{aligned}
 \tag{9.1}$$

where J_1 and J_2 denote the single and double-line currents (See appendix C). This recursion is formally equivalent to the version of the Berends-Giele recursion where the four-point gluon vertex is decomposed into three-point vertices involving a tensor particle [33], apart from the second term in the equation for J_2 , which arises only when vertices of multiplicity five or higher are decomposed into three-point vertices. A tree-level gluon amplitude can now be obtained in a straightforward manner by [33]

$$A_{n+1} = J_1(n) = \text{---} \text{O} .
 \tag{9.2}$$

The interesting and non-trivial outcome of this formulation is that antenna and splitting amplitudes (and also eikonal factors) can be built recursively in the same way as the amplitudes, i.e. , without applying any limiting procedure. For example, an antenna function for $(n - 2)$ unresolved particles corresponds to the n -point double-line current⁸

$$\text{Ant}(1, 2, \dots, n) = V_4 J_2(n) = V_4 \text{=} \text{O} ,
 \tag{9.3}$$

whereas n -point splitting amplitudes are the sum of the n -point single and double-line currents,

$$\text{Split}(1, \dots, n) = J_1(n) + V_c J_2(n) = \text{---} \text{O} + V_c \text{=} \text{O} .
 \tag{9.4}$$

The recursive relations obeyed by the “currents” needed for the full amplitudes and for the splitting functions are formally exactly the same, the only differences being in the definition of vertices and in the initial conditions. In other words, one finds again that the collinear splitting amplitudes enjoy all the known properties of the full amplitudes: not only gauge invariance, dual ward identities, Kleiss-Kuif relations, . . . , but also a fully recursive formulation.

⁸As it was discussed in section 8, it is in fact a sum over all possible double-line currents.

In conclusion, we have shown how to efficiently obtain compact multi-gluon antenna functions. Generalization of our approach to quark antenna functions as well as to the computation of impact factors and Lipatov vertices relevant in the high-energy limit [12] is in progress.

Acknowledgments

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A. Reconstruction functions

In this appendix we present the explicit expressions for the reconstruction functions that have been used. Apart from a difference in the overall sign conventions they are the same as in ref. [20]. They read

$$k_{\hat{a}} = \frac{1}{2(K^2 - t_{1\dots nb})} \left[(1 + \rho)K^2 + 2R \cdot (k_a - k_b - K) + \frac{1}{s_{ab}} G \left(\begin{matrix} k_a, & k_b \\ R, & P_{1,n} \end{matrix} \right) \right] k_a + R$$

$$+ \frac{1}{2(K^2 - t_{a1\dots n})} \left[(1 - \rho)K^2 + 2R \cdot (k_a - k_b - K) + \frac{1}{s_{ab}} G \left(\begin{matrix} k_a, & k_b \\ R, & P_{1,n} \end{matrix} \right) \right] k_b, \quad (\text{A.1})$$

$$k_{\hat{b}} = \frac{1}{2(K^2 - t_{1\dots nb})} \left[(1 - \rho)K^2 + 2\tilde{R} \cdot (k_a - k_b - K) + \frac{1}{s_{ab}} G \left(\begin{matrix} k_a, & k_b \\ \tilde{R}, & P_{1,n} \end{matrix} \right) \right] k_a + \tilde{R}$$

$$+ \frac{1}{2(K^2 - t_{a1\dots n})} \left[(1 + \rho)K^2 + 2\tilde{R} \cdot (k_a - k_b - K) + \frac{1}{s_{ab}} G \left(\begin{matrix} k_a, & k_b \\ \tilde{R}, & P_{1,n} \end{matrix} \right) \right] k_b, \quad (\text{A.2})$$

where $t_{1\dots nb} = (k_1 + \dots + k_n + k_b)^2$, $t_{a1\dots n} = (k_a + k_1 + \dots + k_n)^2$ and $K = k_a + k_1 + \dots + k_n + k_b$. Furthermore

$$R = \sum_{j=1}^n k_j r_j, \quad \tilde{R} = P_{1,n} - R = \sum_{j=1}^n k_j (1 - r_j), \quad (\text{A.3})$$

and

$$\rho = \left[1 + \frac{2G \left(\begin{matrix} a, & R, & b \\ a, & \tilde{R}, & b \end{matrix} \right)}{K^2 s_{ab}^2} + \frac{\Delta(a, R, K, b)}{(K^2)^2 s_{ab}^2} \right]^{\frac{1}{2}}. \quad (\text{A.4})$$

In these formulas, G and Δ denote Gram determinants,

$$G \left(\begin{matrix} p_1, \dots, p_n \\ q_1, \dots, q_n \end{matrix} \right) \equiv \det(2 p_i \cdot q_j), \quad \Delta(p_1, \dots, p_n) \equiv G \left(\begin{matrix} p_1, \dots, p_n \\ p_1, \dots, p_n \end{matrix} \right). \quad (\text{A.5})$$

A suitable choice for the coefficients r_j is [20]

$$r_j = \frac{k_j \cdot (P_{j+1,n} + k_b)}{k_j \cdot K} = \frac{t_{j\dots nb} - t_{(j+1)\dots nb}}{2k_j \cdot K}. \quad (\text{A.6})$$

B. Gluon insertion rule

In this appendix we present the proof of the gluon insertion rule presented in section 4. The idea behind the gluon insertion rule is to start from the Feynman diagrams of the hard amplitude and to see where a soft gluon can be radiated from in such a way that the diagram becomes divergent. There are three places in a Feynman diagram where an additional soft gluon can be emitted from:

- a three-point vertex,
- an internal line,
- an external line.

Let us examine each case separately, and let us start with the situation where a soft gluon is emitted from a three-point vertex (See figure 7). It is easy to see from figure 7a that if a single soft gluon is radiated from a three-point vertex, there is no propagator going on-shell, and thus there is no divergence at all in this situation. But if more than one single gluon is emitted, then there may be a propagator going on-shell, and thus we can get a divergent diagram for more than one single soft gluon emitted from the three-point vertex (See figure 7b and 7c).

Let us first consider the situation where two soft gluons are emitted from the three-point vertex shown in figure 7b. In this situation the propagator goes on-shell, and the rescaling rule, eq. (3.9), tells us that the propagator behaves as t^{-2} . However, as a three-point vertex is proportional to the momenta of the particles, it is easy to see that the second three-point vertex behaves in the soft limit as t , and thus the diagram shown in figure 7b behaves as $t^{-2} \cdot t = t^{-1}$, and so it is not divergent enough to contribute to the soft factor. Let us turn to the situation where three soft gluons are emitted from a three-point vertex (See figure 7c). We get a divergent propagator which behaves as t^{-2} . The four-point vertex however is not divergent at all, and so the diagram shown in figure 7c behaves as t^{-2} , and so it is not divergent enough to contribute to the soft limit. Finally, we come to the conclusion that diagrams where a soft particle is radiated from a three-point vertex do not contribute in the soft limit.

Let's turn to the second case, where the soft particle is emitted from an internal line. Internal lines in a Feynman diagram correspond to off-shell propagators, i.e. , if p is the momentum carried by the internal line, then $p^2 \neq 0$. If a soft particle with momentum k is radiated from this internal line, then we get a propagator of the form $1/(p+k)^2 = 1/(p^2 + 2p \cdot k)$, which stays finite in the soft limit where $k \rightarrow 0$. So there will be no contribution from diagrams where the soft gluon is emitted from an internal line.

In the situation where a soft gluon with momentum k is emitted from an external particle with momentum p , we get a propagator $1/(p+k)^2 = 1/(2p \cdot k)$. In the soft limit,

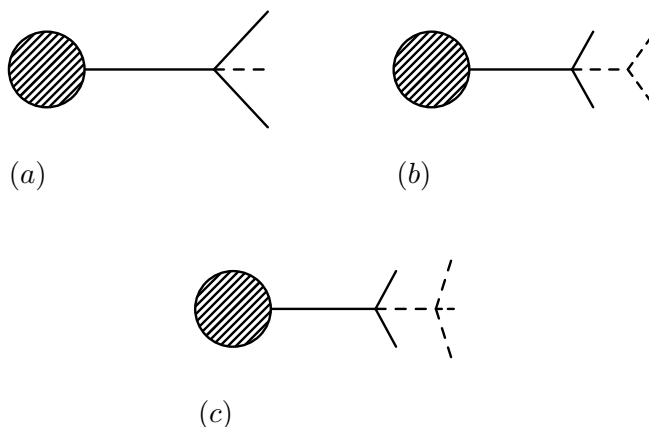


Figure 7: The radiation of a soft gluon from a three-point vertex. Dashed lines indicate soft gluons. The blob represents any subdiagram contributing to this amplitude.

$k \rightarrow 0$, this propagator behaves as $1/t$, and thus has the right divergence to contribute to the soft factor.

Finally, we see that only those Feynman diagrams contribute in the soft limit where the soft gluons are radiated from the external legs of the hard amplitude, which finishes the proof.

C. Recursive formulation of the CSW formalism

Decomposition of the MHV-amplitudes. In this section we present a recursive method to calculate QCD tree-level amplitudes using the CSW construction. This recursive algorithm is equivalent to the recursion presented in ref. [33].⁹ We will start by analysing this construction for MHV-amplitudes. In a second step we generalize this decomposition of MHV-amplitudes to CSW vertices, the fundamental building blocks in the CSW construction.

Let us start with the n -point gluon MHV-amplitude of the form

$$A_n(1^+, 2^+, \dots, (n-2)^+, (n-1)^-, n^-) = \frac{\langle(n-1)n\rangle^4}{\langle 12 \rangle \dots \langle n1 \rangle}. \tag{C.1}$$

This amplitude can be written in terms of eikonal factors

$$A_n(1^+, 2^+, \dots, (n-2)^+, (n-1)^-, n^-) = \frac{\langle(n-1)n\rangle^4}{\langle 1(n-2) \rangle \langle (n-2)(n-1) \rangle \langle (n-1)n \rangle \langle n1 \rangle} \prod_{k=3}^{n-2} D_{1,k}^{k-1}, \tag{C.2}$$

where the eikonal factors are defined by $D_{ij}^k = \frac{\langle ij \rangle}{\langle ik \rangle \langle kj \rangle}$.

Let us consider now the current $J_h(1, \dots, n-1)$ defined as the sum of all CSW diagrams

⁹A numerical study of the complexity of this algorithm can be found in ref. [33]

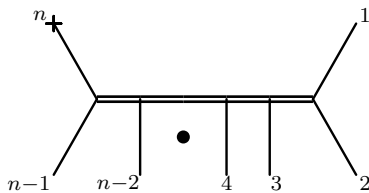


Figure 8: Decomposition of an MHV-amplitude where the MHV vertex has been stretched out into a double-line. The cross indicates the off-shell line.

which have n on-shell external legs and one off-shell leg with helicity h [33].¹⁰ We will refer to such a current as a single-line current. A specific $(n + 1)$ -point amplitude can then be obtained by putting the off-shell leg on-shell,

$$A_n(1, \dots, n, (n + 1)^h) = \lim_{P_{1,n}^2 \rightarrow 0} P_{1,n}^2 J_{-h}(1, \dots, n). \quad (\text{C.3})$$

In particular, for an MHV-type current we can write

$$\begin{aligned} J_+(1^+, 2^+, \dots, (n - 2)^+, (n - 1)^-) &= \frac{1}{P_{1,n-1}^2} \frac{\langle (n - 1), P_{1,n-1} \rangle^4}{\langle 1, 2 \rangle \dots \langle (n - 1), P_{1,n-1} \rangle \langle P_{1,n-1}, 1 \rangle} \\ &= \frac{1}{P_{1,n-1}^2} \frac{\langle (n - 1), P_{1,n-1} \rangle^4}{\langle 1, (n - 2) \rangle \langle (n - 2), (n - 1) \rangle \langle (n - 1), P_{1,n-1} \rangle \langle P_{1,n-1}, 1 \rangle} \prod_{k=3}^{n-2} D_{1k}^{k-1} \\ &= \frac{1}{P_{1,n-1}^2} V_4(1, n - 2, n - 1, P_{1,n-1}) \prod_{k=3}^{n-2} D_{1k}^{k-1}, \end{aligned} \quad (\text{C.4})$$

where we define

$$V_4(a, b, c, d) = \frac{\langle cd \rangle^4}{\langle ab \rangle \langle bc \rangle \langle cd \rangle \langle da \rangle}. \quad (\text{C.5})$$

In figure 8, we introduce a graphical representation for eq. (C.4) in terms of an internal double-line to which the external gluons may couple. We also introduce double-line currents $J_{uv}^m(1, \dots, n)$ defined as the double-line diagram with an external (off-shell) double-line to which the gluons 1 to n are attached. The indices u and v refer to the indices of the first particle one encounters if one follows each line of the double-line into the current, and m refers to the number of negative-helicity gluons attached to the double-line.¹¹ As a double-line is part of an MHV-amplitude with exactly two negative-helicity particles, $J_{uv}^m(1, \dots, n) = 0$ if $m \geq 3$.

The double-line currents contributing to the MHV single-line current $J_+(1^+, \dots, (n - 2)^+, (n - 1)^-)$ can now be easily constructed recursively by adding successively eikonals to

¹⁰In the context of the CSW rules, it makes sense to talk about the helicity of an off-shell particle. Note that, as the off-shell continuation of the spinors involves an arbitrary reference spinor $\eta^{\dot{a}}$, these currents are not gauge invariant objects. However, the $\eta^{\dot{a}}$ dependence drops out in the end [29].

¹¹Up to now these indices are redundant. The double-line currents contributing to an MHV single-line current of the form $J_+(1^+, \dots, (n - 2)^+, (n - 1)^-)$ are all of the form $J_{uv}^m(1^+, \dots, k^+)$ with $u = 1$, $v = k$ and $m = 0$. The meaning of these indices will become clear when we generalize double-line currents to the CSW construction.

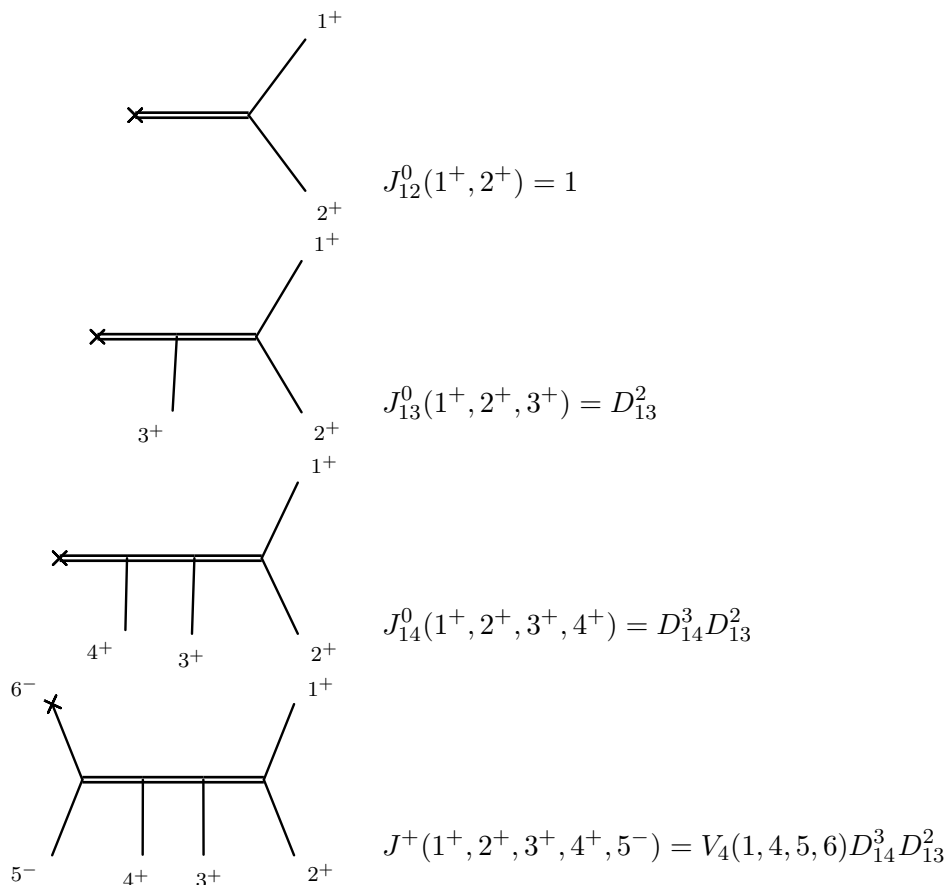


Figure 9: Construction of $A_6(1^+, 2^+, 3^+, 4^+, 5^-, 6^-)$ using the recursion for the double-line current. A cross indicates an off-shell leg.

the double-line current,

$$J_{1k}^0(1^+, \dots, k^+) = D_{1k}^{k-1} J_{1(k-1)}^0(1^+, \dots, (k-1)^+), \quad (\text{C.6})$$

where we define that all one-point double-line currents are zero, and all two-point double-line currents are given by $J_{uv}^0(1^+, 2^+) = \delta_{1u} \delta_{2v}$. The MHV single-line current is then given by

$$J_+(1^+, \dots, (n-2)^+, (n-1)^-) = V_4(1, n-2, n-1, P_{1,n}) J_{1(n-2)}^0(1^+, \dots, (n-2)^+), \quad (\text{C.7})$$

and the MHV-amplitude can then be obtained by putting the off-shell leg on-shell, eq. (C.3). This procedure is illustrated in figure 9 for the six-point MHV-amplitude .

Up to now we have only considered a very special class of MHV-amplitudes, namely those where the two negative-helicity gluons are adjacent in the MHV-amplitude. We now generalize this procedure to an arbitrary MHV-amplitude $A_n(1^+, \dots, i^-, \dots, j^-, \dots, n^+)$. However, the factorisation (C.2) stays no longer true in this case, because an MHV-amplitude only factorises when a soft positive-helicity gluon is radiated. A closer look at the

recursive procedure we have introduced reveals that we only used this method to construct the denominator of the MHV-amplitude, whereas the numerator is completely encoded in $V_4(1, n-2, n-1, n)$. Thus we can generalize this procedure using the following rules:

1. Each time we add a negative-helicity particle j to a double-line current containing already m negative-helicity particles, add a δ_{jm_i} to the double-line current, where $i = 1$ if m passes from 0 to 1, and $i = 2$ if m passes from 1 to 2.
2. At the last step in the recursion sum over all possible values for m_1 and m_2 ,

$$J_h(1, \dots, n-2, n-1) = \sum_{m_1, m_2} V_4(1, n-2, n-1, P_{1,n}; m_1, m_2) J_{1(n-2)}^m(1, \dots, n-2), \quad (\text{C.8})$$

where we defined

$$V_4(a, b, c, d; m_1, m_2) = \frac{\langle m_1, m_2 \rangle^4}{\langle ab \rangle \langle bc \rangle \langle cd \rangle \langle da \rangle}. \quad (\text{C.9})$$

Generalisation to the CSW construction. In this section we extend the recursive relations to the CSW formalism presented in section 2. We start by making the observation that each line in a CSW diagram, as well internal as external, contributing to an n -point amplitude can be uniquely labeled by a multiindex $I = (i_1, i_2)$, with $1 \leq i_1 \leq i_2 \leq n$. Indeed, each external line can be characterized by its momentum p_i , and so we associate to the external particle i the multiindex $I = (i, i)$, and each internal line is characterized by an off-shell momentum $P_{i,j}$, and so we associate to this internal line the multiindex $I = (i, j)$. In general, we have

1. To each line with momentum P_{i_1, i_2} we can associate in a unique way a multiindex $I = (i_1, i_2)$ with $i_1 \leq i_2$ defined by $P_I \equiv P_{i_1, i_2}$.
2. A line in a CSW diagram with multiindex $I = (i_1, i_2)$ is an external line if and only if $i_1 = i_2$.

For later convenience, we introduce the following definitions:

- An on-shell particle with momentum $p_i = P_{i,i}$ is labelled by the multiindex

$$\bar{i} \equiv (i, i). \quad (\text{C.10})$$

- Kronecker-delta:

$$\delta_J^I = \delta_{j_1}^{i_1} \delta_{j_2}^{i_2}. \quad (\text{C.11})$$

- Ordering relation:

$$I < J \Leftrightarrow i_1 \leq i_2 < j_1 \leq j_2. \quad (\text{C.12})$$

We want to write down a recursion relation for the CSW formalism. Looking back to ordinary Feynman rules and the Berends-Giele recursion, it is easy to write out a recursion for the single-line current defined in the previous section [33],

$$J_h(1, \dots, n) = \frac{1}{P_{1,n}^2} \left[\sum_{i=1}^{n-1} A_3 \left(-P_{1,n}^{-h}, P_{1,k}^{h_1}, P_{k+1,n}^{h_2} \right) J_{h_1}(1, \dots, i) J_{h_2}(i+1, \dots, n) \right]$$

$$\begin{aligned}
 & + \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} A_4 \left(-P_{1,n}^{-h}, P_{1,i}^{h_1}, P_{i+1,j}^{h_2}, P_{j+1,n}^{h_3} \right) J_{h_1}(1, \dots, i) \quad (C.13) \\
 & \left. J_{h_2}(i+1, \dots, j) J_{h_3}(j+1, \dots, n) + \dots \right].
 \end{aligned}$$

The dots indicate terms with higher order CSW vertices. A sum over the helicities (h, h_1, h_2, \dots) with $-h + h_1 + h_2 + \dots = n - 4$ is implicitly understood. According to the CSW rules, the vertices A_n correspond to off-shell continued n -point MHV-amplitudes, and so these vertices can themselves be constructed recursively using the recursive procedure introduced in the previous section. Merging the recursion relations (C.6)–(C.7) and (C.13) and replacing indices by multiindices, we obtain a recursion for the single and double-line currents,

$$\begin{aligned}
 J_{UV}^m(1, \dots, n) & = \delta_{v_1-1}^{u_2} \delta_{(1,u_2)}^U \delta_{(v_1,n)}^V \epsilon_m^{h_1 h_2}(P_U, P_V) J_{h_1}(1, \dots, u_2) J_{h_2}(v_1, \dots, n) \\
 & + (1 - \delta_{v_1-1}^{u_2}) \sum_{\substack{W \\ w_2=v_1-1}} \delta_{(v_1,n)}^V \epsilon_{mm'}^h(P_V) D_{UW}^V J_{UW}^{m'}(1, \dots, v_1 - 1) \quad (C.14) \\
 & J_h(v_1, \dots, n),
 \end{aligned}$$

$$\begin{aligned}
 J_h(1, \dots, n) & = \frac{1}{P_{1,n}^2} \sum_{k=1}^{n-1} \sum_{M_1, M_2} \left[\epsilon^{h h_1 h_2}(P_{1,n}, P_{1,k}, P_{k+1,n}) V_3(P_{1,k}, P_{k+1,n}; M_1, M_2) \right. \\
 & J_{h_1}(1, \dots, k) J_{h_2}(k+1, \dots, n) \quad (C.15) \\
 & + \sum_{\substack{U < V \\ v_2=k}} \epsilon_{|2-m|}^{h h_1}(P_{1,n}, P_{k+1,n}) V_4(P_U, P_V, P_{k+1,n}, P_{1,n}; P_{M_1}, P_{M_2}) \\
 & \left. J_{UV}^m(1, \dots, k) J_{h_1}(k+1, \dots, n) \right],
 \end{aligned}$$

where a sum over all repeated helicity indices is understood. Capital letters indicate multiindices and the vertices are summarized in table 1. A graphical representation of the recursion can be found in figure 10. V_3 and V_4 correspond to the three and four-point CSW vertices,

$$V_3(A, B, C; M_1, M_2) = \frac{\langle M_1, M_2 \rangle^4}{\langle AB \rangle \langle BC \rangle \langle CA \rangle}, \quad (C.16)$$

$$V_4(A, B, C, D; M_1, M_2) = \frac{\langle M_1, M_2 \rangle^4}{\langle AB \rangle \langle BC \rangle \langle CD \rangle \langle DA \rangle}, \quad (C.17)$$

and the epsilon functions appearing in the recursion keep track of the helicities,

$$\epsilon_{mm'}^h(I) = \begin{cases} \delta_I^{M_m} & , \text{if } h = -1, m = m' + 1, m \leq 2 \\ 0 & \text{else} \end{cases} \quad (C.18)$$

$$\epsilon_m^{h_I h_J}(I, J) = \begin{cases} \delta_I^{M_1} \delta_J^{M_2} & , \text{if } h_I = h_J = -1, m = 2, \\ \delta_I^{M_1} & , \text{if } h_I = -h_J = -1, m = 1, \\ \delta_J^{M_1} & , \text{if } h_I = -h_J = 1, m = 1, \\ 1 & , \text{if } h_I = h_J = 1, m = 0, \\ 0 & , \text{else,} \end{cases} \quad (C.19)$$

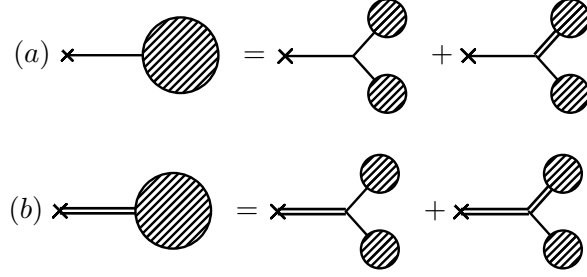


Figure 10: Recursion for the single-line currents (a) and the double-line currents (b). A cross indicates an off-shell line.

Vertex	Vertex factor
	$\epsilon^{h_K h_I h_J}(I, J, K) \frac{\langle M_1 M_2 \rangle^4}{\langle IJ \rangle \langle JK \rangle \langle KI \rangle}$
	$\epsilon_m^{h_I h_J}(U, V) \delta_U^I \delta_V^J$
	$\epsilon_{mm'}^h(I) D_{UV}^I \delta_U^{U'} \delta_V^I$
	$\epsilon_{ 2-m }^{h_J h_K}(J, K) \frac{\langle M_1 M_2 \rangle^4}{\langle UV \rangle \langle VJ \rangle \langle JK \rangle \langle JK \rangle}$

Table 1: Vertices appearing in the recursion for the single and double-line currents.

$$\epsilon^{h_I h_J h_K}(I, J, K) = \begin{cases} \delta_X^{M_1} \delta_Y^{M_2} & \text{if } h_X = h_Y = -1, h_1 + h_2 + h_3 = -1, \\ 0 & \text{, else .} \end{cases} \quad (\text{C.20})$$

D. Recursive relations for soft factors

In this appendix we discuss how to build recursive relations for soft factors out of the recursion for antenna functions, eq. (8.1), in a similar way as we did for the splitting functions in section 8. We start by giving some general considerations about the off-shell continuation that appears in the CSW formalism in the soft limit. In general, a propagator

involving more than two particles can be written as

$$P_{i,j}^2 = \sum_{\substack{k,l=i \\ k < l}}^j s_{kl}. \quad (\text{D.1})$$

In particular, if i is a hard particle, say $i = a$, and all other momenta are soft, we can write

$$P_{a,j}^2 = \sum_{l=s_1}^j s_{al} + \sum_{k=s_1}^{j-1} \sum_{l=s_2}^j s_{kl}. \quad (\text{D.2})$$

Using the power counting (3.9), it is easy to see that the second term goes to zero much faster than the first one, so we have the following behavior in the soft limit

$$P_{a,j}^2 \sim \sum_{l=s_1}^j s_{al}. \quad (\text{D.3})$$

Similar arguments hold true for spinor products involving off-shell momenta in the CSW formalism,

$$\langle k, P_{i,j} \rangle = \langle k | P_{i,j} | \eta \rangle = \sum_{l=i}^j \langle kl \rangle [l\eta]. \quad (\text{D.4})$$

Following exactly the same lines as for the propagators, we can derive the following rule for off-shell continued spinor products in the soft limit,

Rule 5 (Off-shell continuation in the soft limit) *In the soft limit, each spinor product of the form $\langle k, P_{a,j} \rangle$ has to be interpreted as*

$$\langle k P_{a,j} \rangle \rightarrow \langle ka \rangle [a\eta],$$

except for $k = a$, because in this case we have trivially $\langle a, P_{a,j} \rangle = \langle a, P_{1,j} \rangle$, and equivalently for b .

In particular, this implies the following rules for off-shell continued momenta

$$\begin{aligned} \langle a, P_{j,b} \rangle &\rightarrow \langle ab \rangle [b\eta], \\ \langle P_{a,j}, b \rangle &\rightarrow \langle ab \rangle [a\eta], \\ \langle P_{i,j}, P_{k,b} \rangle &\rightarrow \langle P_{i,j}, b \rangle [b\eta], \\ \langle P_{a,i}, P_{j,k} \rangle &\rightarrow \langle a, P_{j,k} \rangle [a\eta], \\ \langle P_{a,j}, P_{k,b} \rangle &\rightarrow \langle ab \rangle [a, \eta] [b\eta]. \end{aligned} \quad (\text{D.5})$$

Let us turn now to the recursive relations. We know that in the limit where all the particles are soft, we have

$$\text{Ant}(\hat{a}^{h_{\hat{a}}}, \hat{b}^{h_{\hat{b}}} \leftarrow a, 1, \dots, n, b) \longrightarrow \text{Soft}(a, 1, \dots, n, b). \quad (\text{D.6})$$

As the soft factor is independent of the helicities of the reference particles a and b , some antenna functions are not divergent in this limit, but only those are divergent where $h_{\hat{a}} =$

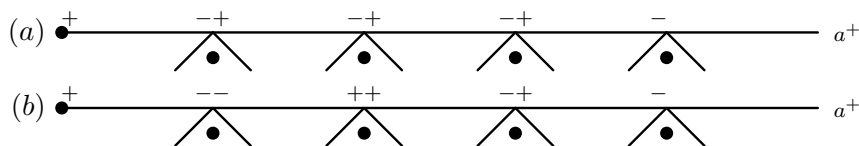


Figure 11: Diagrams contributing to $J_+(a^+, 1, \dots, k)$ in the soft limit.

$-h_a$ and $h_{\hat{b}} = -h_b$. Taking the soft limit of eq. (8.1), we can derive a formula for soft factors in terms of single and double-line currents. We will show this procedure explicitly for $\text{Ant}(\hat{a}^-, \hat{b}^- \leftarrow a^+, 1, \dots, n, b^+)$. Applying the rules given in section 3, it is easy to see that in the soft limit

$$V_4(U, V, \hat{a}, \hat{b}; \hat{a}, \hat{b}) \rightarrow V_s(a, b, \eta) = \begin{cases} 1 & , \text{if } U = (a, a) \text{ and } V = (b, b), \\ 1/[a\eta]^2 & , \text{if } U \neq (a, a) \text{ and } V = (b, b), \\ 1/[b\eta]^2 & , \text{if } U = (a, a) \text{ and } V \neq (b, b), \\ 1/([a\eta]^2[b\eta]^2) & , \text{if } U \neq (a, a) \text{ and } V \neq (n, n). \end{cases} \quad (\text{D.7})$$

Finally, the contribution from an antenna function to a soft factor is

$$\sum_{\substack{U < V \\ v_2 = b}} V_s(a, b, \eta) J_{UV}^0(a^+, 1, \dots, n, b^+). \quad (\text{D.8})$$

However, eq. (D.8) generates not only diagrams that contribute to the soft limit, but also subleading diagrams. To see this, consider the diagrams contained in $J_+(a^+, 1, \dots, k)$ shown in figure 11. Let us start with the situation where all external lines in figure 11 are on-shell (except the off-shell line (a, k) indicated with a dot). Using the power counting rules (3.9), it follows that the denominators of both diagrams shown in figure 11 behave as $1/t^{2k+4k_-}$, where k_- is the number of negative helicities in the set $\{1, \dots, k\}$. We then come to the following conclusion:

- For diagrams of the form shown in figure 11.a, the numerator behaves as t^{4k_-} , and so these diagrams behave as $t^{4k_-}/t^{2k+4k_-} = 1/t^{2k}$, which is exactly the divergence we want.
- For diagrams of the form shown in figure 11.b, the numerator behaves as t^{4k_-+8} , and so these diagrams behave as $t^{4k_-+8}/t^{2k+4k_-} = 1/t^{2k-8}$, and so these diagrams are not divergent enough to contribute to the soft factor.

Up to now we have only considered diagrams in figure 11 where all external lines are on-shell. Let us turn to the situation where some of these lines may themselves be off-shell, i.e. , they correspond to single-line currents $J_h(i_1, \dots, i_2)$. First, a simple power counting argument shows that if the particles i_1, \dots, i_2 are soft,

$$J_+(i_1, \dots, i_2) \sim 1/t^{2(i_2-i_1+1)-4}, \quad J_-(i_1, \dots, i_2) \sim 1/t^{2(i_2-i_1+1)}. \quad (\text{D.9})$$

Using this result, it is easy to show that

- Replacing a positive-helicity particle i by a single-line current $J_+(i_1, \dots, i_2)$ amounts to the replacement

$$\frac{1}{\langle Xi \rangle \langle iY \rangle} \rightarrow \frac{1}{\langle XI \rangle \langle IY \rangle} J_+(i_1, \dots, i_2). \quad (D.10)$$

$$\sim 1/t^2 \qquad \sim 1/t^4 \ 1/t^{2(i_2-i_1+1)-4}$$

So the divergence in $1/t^2$ of the soft particle i gets replaced by the divergence in $1/t^{2(i_2-i_1+1)}$ of the soft particles i_1, \dots, i_2 .

- Replacing a negative-helicity particle i by a single-line current $J_-(i_1, \dots, i_2)$ amounts to the replacement

$$\frac{\langle Zi \rangle^4}{\langle Xi \rangle \langle iY \rangle} \rightarrow \frac{\langle ZI \rangle^4}{\langle XI \rangle \langle IY \rangle} J_-(i_1, \dots, i_2). \quad (D.11)$$

$$\sim t^2 \qquad \sim t^4/t^{2(i_2-i_1+1)}$$

Furthermore, there must be somewhere in the diagram a $1/t^4$ factor knocking the t^2 on the left-hand side down to a divergence in $1/t^2$ for the soft particle i . On the right-hand side, this same term now kills the t^4 in the numerator, giving the divergence in $1/t^{2(j_2-j_1+1)}$ for the soft particles i_1, \dots, i_2 .

So we come to the conclusion that only the diagrams in $J_+(a^+, 1 \dots, k)$ of the form shown in figure 11.a contribute in the soft limit. These diagrams are exactly those which do not contain any subcurrent of the form $J_-(a^+, 1, \dots, l)$. A similar result holds of course for b , i.e. , only those diagrams contribute to the soft limit which do not contain a subcurrent of the form $J_-(l, \dots, n, b^+)$. Finally we can write

$$\text{Soft}(a, 1, \dots, n, b) = \sum_{\substack{U < V \\ v_2 = b}} V_s(a, b, \eta) \bar{J}_{UV}^0(a^+, 1, \dots, n, b^+), \quad (D.12)$$

where $\bar{J}_{UV}^0(a^+, 1, \dots, n, b^+)$ denotes a double-line current which does not contain any subcurrent of the form $J_-(a^+, 1, \dots, l)$ or $J_-(l, \dots, n, b^+)$.

Of course we could have used a different antenna function to extract the soft factors. In general, we get the result

$$\text{Soft}(a, 1, \dots, n, b) = \sum_{\substack{U < V \\ v_2 = b}} V_s(a, b, \eta) \epsilon^{h_a h_b} \bar{J}_{UV}^{(2-h_a-h_b)/2}(a^{h_a}, 1, \dots, n, b^{h_b}), \quad (D.13)$$

where $\bar{J}_{UV}^0(a^{h_a}, 1, \dots, n, b^{h_b})$ is the double-line current which does not contain any subcurrents of the form $J_{-h_a}(a^{h_a}, 1, \dots, l)$ or $J_{-h_b}(l, \dots, n, b^{h_b})$, and

$$\begin{aligned} \epsilon^{++} &= 1, & \epsilon^{+-} &= \frac{\langle aM \rangle^4}{\langle ab \rangle^4}, \\ \epsilon^{+0} &= \frac{\langle Mb \rangle^4}{\langle ab \rangle^4}, & \epsilon^{--} &= \frac{\langle M_1 M_2 \rangle^4}{\langle ab \rangle^4}. \end{aligned} \quad (D.14)$$

We calculated numerically the soft factors for one, two or three soft particles, and we checked numerically that these quantities are independent of the arbitrary spinor η as well as of the helicities of the spectator particles a and b .

E. Explicit results

In this appendix we present our results for all independent NLO, NNLO and NNNLO antenna functions. These results are obtained from the general formulas in section 6. We give explicit results for antenna functions of the form $\text{Ant}(\hat{a}^-, \hat{b}^- \leftarrow a, 1, \dots, n, b)$ and $\text{Ant}(\hat{a}^-, \hat{b}^+ \leftarrow a, 1, \dots, n, b)$. The remaining antenna functions are related to the previous ones by parity and reflection identity (6.1). At NNNLO we have to include also antenna functions of the form $\text{Ant}(\hat{a}^+, \hat{b}^+ \leftarrow a, 1, \dots, n, b)$.

NLO antenna functions. All NLO antenna functions are just MHV-type antenna functions.

$$\begin{aligned}
 \text{Ant}(\hat{a}^-, \hat{b}^- \leftarrow a^+, 1^+, b^+) &= \frac{\langle \hat{a}\hat{b} \rangle^3}{\langle a1 \rangle \langle 1b \rangle \langle b\hat{a} \rangle \langle \hat{b}a \rangle}. \\
 \text{Ant}(\hat{a}^-, \hat{b}^- \leftarrow a^-, 1^+, b^+) &= -\frac{[1b]^3}{[a1][b\hat{a}][\hat{a}\hat{b}][\hat{b}a]}. \\
 \text{Ant}(\hat{a}^-, \hat{b}^- \leftarrow a^+, 1^-, b^+) &= -\frac{[ab]^4}{[a1][1b][b\hat{a}][\hat{a}\hat{b}][\hat{b}a]}. \\
 \text{Ant}(\hat{a}^-, \hat{b}^- \leftarrow a^+, 1^+, b^-) &= -\frac{[a1]^3}{[1b][b\hat{a}][\hat{a}\hat{b}][\hat{b}a]}. \\
 \text{Ant}(\hat{a}^-, \hat{b}^- \leftarrow a^-, 1^-, b^+) &= 0. \\
 \text{Ant}(\hat{a}^-, \hat{b}^- \leftarrow a^-, 1^+, b^-) &= 0. \\
 \text{Ant}(\hat{a}^-, \hat{b}^- \leftarrow a^+, 1^-, b^-) &= 0. \\
 \text{Ant}(\hat{a}^-, \hat{b}^- \leftarrow a^-, 1^-, b^-) &= 0. \\
 \text{Ant}(\hat{a}^-, \hat{b}^+ \leftarrow a^+, 1^+, b^+) &= 0. \\
 \text{Ant}(\hat{a}^-, \hat{b}^+ \leftarrow a^-, 1^+, b^+) &= 0. \\
 \text{Ant}(\hat{a}^-, \hat{b}^+ \leftarrow a^+, 1^-, b^+) &= \frac{\langle \hat{a}1 \rangle^4}{\langle a1 \rangle \langle 1b \rangle \langle b\hat{a} \rangle \langle \hat{a}\hat{b} \rangle \langle \hat{b}a \rangle}. \\
 \text{Ant}(\hat{a}^-, \hat{b}^+ \leftarrow a^+, 1^+, b^-) &= \frac{\langle b\hat{a} \rangle^3}{\langle a1 \rangle \langle 1b \rangle \langle \hat{a}\hat{b} \rangle \langle \hat{b}a \rangle}. \\
 \text{Ant}(\hat{a}^-, \hat{b}^+ \leftarrow a^-, 1^-, b^+) &= 0. \\
 \text{Ant}(\hat{a}^-, \hat{b}^+ \leftarrow a^-, 1^+, b^-) &= -\frac{[\hat{a}1]^4}{[a1][1b][b\hat{a}][\hat{a}\hat{b}][\hat{b}a]}. \\
 \text{Ant}(\hat{a}^-, \hat{b}^+ \leftarrow a^+, 1^-, b^-) &= -\frac{[b\hat{a}]^3}{[a1][1b][\hat{a}\hat{b}][\hat{b}a]}. \\
 \text{Ant}(\hat{a}^-, \hat{b}^+ \leftarrow a^-, 1^-, b^-) &= 0.
 \end{aligned}$$

NNLO antenna functions. All NNLO antenna functions are just MHV and NMHV-type antenna functions.

$$\text{Ant}(\hat{a}^-, \hat{b}^- \leftarrow a^+, 1^+, 2^+, b^+) = \frac{\langle \hat{a}\hat{b} \rangle^3}{\langle a1 \rangle \langle 12 \rangle \langle 2b \rangle \langle b\hat{a} \rangle \langle \hat{b}a \rangle}.$$

$$\begin{aligned}
\text{Ant}(\hat{a}^-, \hat{b}^- \leftarrow a^-, 1^-, 2^+, b^+) &= \frac{[2b]^3}{[a1][12][b\hat{a}][\hat{a}\hat{b}][\hat{b}a]}. \\
\text{Ant}(\hat{a}^-, \hat{b}^- \leftarrow a^-, 1^+, 2^-, b^+) &= \frac{[1b]^4}{[a1][12][2b][b\hat{a}][\hat{a}\hat{b}][\hat{b}a]}. \\
\text{Ant}(\hat{a}^-, \hat{b}^- \leftarrow a^-, 1^+, 2^+, b^-) &= \frac{[12]^3}{[a1][2b][b\hat{a}][\hat{a}\hat{b}][\hat{b}a]}. \\
\text{Ant}(\hat{a}^-, \hat{b}^- \leftarrow a^+, 1^-, 2^-, b^+) &= \frac{[ab]^4}{[a1][12][2b][b\hat{a}][\hat{a}\hat{b}][\hat{b}a]}. \\
\text{Ant}(\hat{a}^-, \hat{b}^- \leftarrow a^+, 1^-, 2^+, b^-) &= \frac{[a2]^4}{[a1][12][2b][b\hat{a}][\hat{a}\hat{b}][\hat{b}a]}. \\
\text{Ant}(\hat{a}^-, \hat{b}^- \leftarrow a^+, 1^+, 2^-, b^-) &= \frac{[a1]^3}{[12][2b][b\hat{a}][\hat{a}\hat{b}][\hat{b}a]}. \\
\text{Ant}(\hat{a}^-, \hat{b}^- \leftarrow a^-, 1^-, 2^-, b^+) &= 0. \\
\text{Ant}(\hat{a}^-, \hat{b}^- \leftarrow a^-, 1^-, 2^+, b^-) &= 0. \\
\text{Ant}(\hat{a}^-, \hat{b}^- \leftarrow a^-, 1^+, 2^-, b^-) &= 0. \\
\text{Ant}(\hat{a}^-, \hat{b}^- \leftarrow a^+, 1^-, 2^-, b^-) &= 0. \\
\text{Ant}(\hat{a}^-, \hat{b}^- \leftarrow a^-, 1^-, 2^-, b^-) &= 0.
\end{aligned}$$

$$\begin{aligned}
\text{Ant}(\hat{a}^-, \hat{b}^+ \leftarrow a^+, 1^+, 2^+, b^+) &= 0. \\
\text{Ant}(\hat{a}^-, \hat{b}^+ \leftarrow a^-, 1^+, 2^+, b^+) &= 0. \\
\text{Ant}(\hat{a}^-, \hat{b}^+ \leftarrow a^+, 1^-, 2^+, b^+) &= \frac{\langle \hat{a}1 \rangle^4}{\langle a1 \rangle \langle 12 \rangle \langle 2b \rangle \langle b\hat{a} \rangle \langle \hat{a}\hat{b} \rangle \langle \hat{b}a \rangle}. \\
\text{Ant}(\hat{a}^-, \hat{b}^+ \leftarrow a^+, 1^+, 2^-, b^+) &= \frac{\langle \hat{a}2 \rangle^4}{\langle a1 \rangle \langle 12 \rangle \langle 2b \rangle \langle b\hat{a} \rangle \langle \hat{a}\hat{b} \rangle \langle \hat{b}a \rangle}. \\
\text{Ant}(\hat{a}^-, \hat{b}^+ \leftarrow a^+, 1^+, 2^+, b^-) &= \frac{\langle b\hat{a} \rangle^3}{\langle a1 \rangle \langle 12 \rangle \langle 2b \rangle \langle \hat{a}\hat{b} \rangle \langle \hat{b}a \rangle}. \\
\text{Ant}(\hat{a}^-, \hat{b}^+ \leftarrow a^-, 1^-, 2^-, b^+) &= 0. \\
\text{Ant}(\hat{a}^-, \hat{b}^+ \leftarrow a^-, 1^-, 2^+, b^-) &= \frac{[2\hat{b}]^4}{[a1][12][2b][b\hat{a}][\hat{a}\hat{b}][\hat{b}a]}. \\
\text{Ant}(\hat{a}^-, \hat{b}^+ \leftarrow a^-, 1^+, 2^-, b^-) &= \frac{[1\hat{b}]^4}{[a1][12][2b][b\hat{a}][\hat{a}\hat{b}][\hat{b}a]}. \\
\text{Ant}(\hat{a}^-, \hat{b}^+ \leftarrow a^+, 1^-, 2^-, b^-) &= \frac{[\hat{b}a]^4}{[a1][12][2b][b\hat{a}][\hat{a}\hat{b}]}. \\
\text{Ant}(\hat{a}^-, \hat{b}^+ \leftarrow a^-, 1^-, 2^-, b^-) &= 0.
\end{aligned}$$

$$\text{Ant}(\hat{a}^-, \hat{b}^- \leftarrow a^-, 1^+, 2^+, b^+) = \frac{\langle \hat{a}\hat{b} \rangle^3}{\langle \hat{a}\hat{b} \rangle \langle 2|a+1|\eta \rangle}$$

$$\begin{aligned}
 & \left(-\frac{\langle a|1+2|\eta\rangle^3}{s_{a12}\langle 12\rangle\langle a1\rangle\langle b|a+1+2|\eta\rangle\langle \hat{b}|a+1+2|\eta\rangle} - \frac{[1\eta]^3}{\langle 2b\rangle\langle \hat{b}|a+1|\eta\rangle[a1][a\eta]} \right) \\
 \text{Ant}(\hat{a}^-, \hat{b}^- \leftarrow a^+, 1^-, 2^+, b^+) &= -\frac{\langle \hat{a}\hat{b}\rangle^3\langle 1|2+b|\eta\rangle^3}{s_{b12}\langle 12\rangle\langle 2b\rangle\langle \hat{a}\hat{b}\rangle\langle b|1+2|\eta\rangle\langle a|1+2+b|\eta\rangle\langle \hat{a}|1+2+b|\eta\rangle} \\
 & - \frac{\langle \hat{a}\hat{b}\rangle^3\langle 1|a+2|\eta\rangle^4}{s_{a12}\langle 12\rangle\langle a1\rangle\langle \hat{a}\hat{b}\rangle\langle 2|a+1|\eta\rangle\langle a|1+2|\eta\rangle\langle b|a+1+2|\eta\rangle\langle \hat{b}|a+1+2|\eta\rangle} \\
 & + \frac{\langle \hat{a}\hat{b}\rangle^3[2\eta]^3}{\langle \hat{a}\hat{b}\rangle\langle \hat{a}\hat{b}\rangle\langle a|1+2|\eta\rangle\langle b|1+2|\eta\rangle[12][1\eta]} \\
 & - \frac{\langle \hat{a}\hat{b}\rangle^3[a\eta]^3}{\langle 2b\rangle\langle \hat{a}\hat{b}\rangle\langle 2|a+1|\eta\rangle\langle \hat{b}|a+1|\eta\rangle[1\eta][a1]} \\
 \text{Ant}(\hat{a}^-, \hat{b}^- \leftarrow a^+, 1^+, 2^-, b^+) &= \text{Ant}(\hat{b}^-, \hat{a}^- \leftarrow b^+, 2^-, 1^+, a^+). \\
 \text{Ant}(\hat{a}^-, \hat{b}^- \leftarrow a^+, 1^+, 2^+, b^-) &= \text{Ant}(\hat{b}^-, \hat{a}^- \leftarrow b^-, 2^+, 1^+, a^+). \\
 \text{Ant}(\hat{a}^-, \hat{b}^+ \leftarrow a^-, 1^-, 2^+, b^+) &= 0. \\
 \text{Ant}(\hat{a}^-, \hat{b}^+ \leftarrow a^-, 1^+, 2^-, b^+) &= -\frac{1}{\langle \hat{a}\hat{b}\rangle\langle \hat{a}\hat{b}\rangle\langle 2|a+1|\eta\rangle} \\
 & \left(\frac{\langle a2\rangle^4\langle \hat{a}|1+2|\eta\rangle^4}{s_{a12}\langle 12\rangle\langle a1\rangle\langle a|1+2|\eta\rangle\langle b|a+1+2|\eta\rangle\langle \hat{b}|a+1+2|\eta\rangle} + \frac{\langle \hat{a}2\rangle^4[1\eta]^3}{\langle 2b\rangle\langle \hat{b}|a+1|\eta\rangle[a1][a\eta]} \right) \\
 \text{Ant}(\hat{a}^-, \hat{b}^+ \leftarrow a^-, 1^+, 2^+, b^-) &= \frac{\langle \hat{a}\hat{b}\rangle^3}{\langle \hat{a}\hat{b}\rangle\langle 2|a+1|\eta\rangle} \\
 & \left(-\frac{\langle a|1+2|\eta\rangle^3}{s_{a12}\langle 12\rangle\langle a1\rangle\langle b|a+1+2|\eta\rangle\langle \hat{b}|a+1+2|\eta\rangle} - \frac{[1\eta]^3}{\langle 2b\rangle\langle \hat{b}|a+1|\eta\rangle[a1][a\eta]} \right) \\
 \text{Ant}(\hat{a}^-, \hat{b}^+ \leftarrow a^+, 1^-, 2^-, b^+) &= -\frac{\langle 12\rangle^3\langle \hat{a}|1+2+b|\eta\rangle^3}{s_{b12}\langle 2b\rangle\langle \hat{a}\hat{b}\rangle\langle \hat{a}\hat{b}\rangle\langle 1|2+b|\eta\rangle\langle b|1+2|\eta\rangle\langle a|1+2+b|\eta\rangle} \\
 & - \frac{\langle 12\rangle^3\langle \hat{a}|1+2|\eta\rangle^4}{s_{a12}\langle a1\rangle\langle \hat{a}\hat{b}\rangle\langle \hat{a}\hat{b}\rangle\langle 2|a+1|\eta\rangle\langle a|1+2|\eta\rangle\langle b|a+1+2|\eta\rangle\langle \hat{b}|a+1+2|\eta\rangle} \\
 & + \frac{\langle \hat{a}\hat{b}\rangle\langle \hat{a}\hat{b}\rangle\langle \hat{a}\hat{b}\rangle\langle a|1+2|\eta\rangle\langle b|1+2|\eta\rangle[12][1\eta][2\eta]}{\langle \hat{a}|1+2|\eta\rangle^4} \\
 & - \frac{\langle \hat{a}2\rangle^4[a\eta]^3}{\langle 2b\rangle\langle \hat{a}\hat{b}\rangle\langle \hat{a}\hat{b}\rangle\langle 2|a+1|\eta\rangle\langle \hat{b}|a+1|\eta\rangle[1\eta][a1]} \\
 & - \frac{\langle \hat{a}1\rangle^4[b\eta]^3}{\langle a1\rangle\langle \hat{a}\hat{b}\rangle\langle \hat{a}\hat{b}\rangle\langle 1|2+b|\eta\rangle\langle \hat{a}|2+b|\eta\rangle[2b][2\eta]} \\
 \text{Ant}(\hat{a}^-, \hat{b}^+ \leftarrow a^+, 1^-, 2^+, b^-) &= -\frac{\langle 1b\rangle^4\langle \hat{a}|1+2+b|\eta\rangle^3}{s_{b12}\langle 12\rangle\langle 2b\rangle\langle \hat{a}\hat{b}\rangle\langle \hat{a}\hat{b}\rangle\langle 1|2+b|\eta\rangle\langle b|1+2|\eta\rangle\langle a|1+2+b|\eta\rangle} \\
 & - \frac{\langle \hat{a}\hat{b}\rangle^3\langle 1|a+2|\eta\rangle^4}{s_{a12}\langle 12\rangle\langle a1\rangle\langle \hat{a}\hat{b}\rangle\langle 2|a+1|\eta\rangle\langle a|1+2|\eta\rangle\langle b|a+1+2|\eta\rangle\langle \hat{b}|a+1+2|\eta\rangle} \\
 & + \frac{\langle \hat{a}\hat{b}\rangle^3[2\eta]^3}{\langle \hat{a}\hat{b}\rangle\langle \hat{a}\hat{b}\rangle\langle a|1+2|\eta\rangle\langle b|1+2|\eta\rangle[12][1\eta]} - \frac{\langle \hat{a}\hat{b}\rangle^3[a\eta]^3}{\langle 2b\rangle\langle \hat{a}\hat{b}\rangle\langle 2|a+1|\eta\rangle\langle \hat{b}|a+1|\eta\rangle[1\eta][a1]}
 \end{aligned}$$

$$\begin{aligned}
 & - \frac{\langle \hat{a}1 \rangle^4 [2\eta]^3}{\langle a1 \rangle \langle \hat{a}\hat{b} \rangle \langle \hat{a}\hat{b} \rangle \langle 1|2+b|\eta \rangle \langle \hat{a}|2+b|\eta \rangle [2b][b\eta]} \\
 \text{Ant}(\hat{a}^-, \hat{b}^+ \leftarrow a^+, 1^+, 2^-, b^-) &= - \frac{\langle 2b \rangle^3 \langle \hat{a}|1+2+b|\eta \rangle^3}{s_{b12} \langle 12 \rangle \langle \hat{a}\hat{b} \rangle \langle \hat{a}\hat{b} \rangle \langle 1|2+b|\eta \rangle \langle b|1+2|\eta \rangle \langle a|1+2+b|\eta \rangle} \\
 & - \frac{\langle \hat{a}\hat{b} \rangle^3 \langle 2|a+1|\eta \rangle^3}{s_{a12} \langle 12 \rangle \langle a1 \rangle \langle \hat{a}\hat{b} \rangle \langle a|1+2|\eta \rangle \langle b|a+1+2|\eta \rangle \langle \hat{b}|a+1+2|\eta \rangle} \\
 & + \frac{\langle \hat{a}\hat{b} \rangle^3 [1\eta]^3}{\langle \hat{a}\hat{b} \rangle \langle \hat{a}\hat{b} \rangle \langle a|1+2|\eta \rangle \langle b|1+2|\eta \rangle [12][2\eta]} - \frac{\langle \hat{a}|2+b|\eta \rangle^3}{\langle a1 \rangle \langle \hat{a}\hat{b} \rangle \langle \hat{a}\hat{b} \rangle \langle 1|2+b|\eta \rangle [2b][2\eta][b\eta]}
 \end{aligned}$$

NNNLO antenna functions. All NNNLO antenna functions are just MHV and NMHV-type antenna functions.

$$\text{Ant}(\hat{a}^-, \hat{b}^- \leftarrow a^-, 1^-, 2^-, 3^-, b^-) = 0.$$

$$\text{Ant}(\hat{a}^-, \hat{b}^- \leftarrow a^-, 1^-, 2^-, 3^-, b^+) = 0.$$

$$\text{Ant}(\hat{a}^-, \hat{b}^- \leftarrow a^-, 1^-, 2^-, 3^+, b^-) = 0.$$

$$\text{Ant}(\hat{a}^-, \hat{b}^- \leftarrow a^-, 1^-, 2^+, 3^-, b^-) = 0.$$

$$\text{Ant}(\hat{a}^-, \hat{b}^- \leftarrow a^-, 1^+, 2^-, 3^-, b^-) = 0.$$

$$\text{Ant}(\hat{a}^-, \hat{b}^- \leftarrow a^+, 1^-, 2^-, 3^-, b^-) = 0.$$

$$\text{Ant}(\hat{a}^-, \hat{b}^- \leftarrow a^+, 1^+, 2^+, 3^+, b^+) = \frac{\langle \hat{a}\hat{b} \rangle^3}{\langle a1 \rangle \langle 12 \rangle \langle 23 \rangle \langle 3b \rangle \langle b\hat{a} \rangle \langle \hat{b}a \rangle}.$$

$$\text{Ant}(\hat{a}^-, \hat{b}^- \leftarrow a^-, 1^-, 2^-, 3^+, b^+) = \frac{-[3b]^3}{[a1][12][23][b\hat{a}][\hat{a}\hat{b}][\hat{b}a]}.$$

$$\text{Ant}(\hat{a}^-, \hat{b}^- \leftarrow a^-, 1^-, 2^+, 3^-, b^+) = \frac{-[2b]^4}{[a1][12][23][3b][b\hat{a}][\hat{a}\hat{b}][\hat{b}a]}.$$

$$\text{Ant}(\hat{a}^-, \hat{b}^- \leftarrow a^-, 1^+, 2^-, 3^-, b^+) = \frac{-[1b]^4}{[a1][12][23][3b][b\hat{a}][\hat{a}\hat{b}][\hat{b}a]}.$$

$$\text{Ant}(\hat{a}^-, \hat{b}^- \leftarrow a^+, 1^-, 2^-, 3^-, b^+) = \frac{-[ab]^4}{[a1][12][23][3b][b\hat{a}][\hat{a}\hat{b}][\hat{b}a]}.$$

$$\text{Ant}(\hat{a}^-, \hat{b}^- \leftarrow a^-, 1^-, 2^+, 3^+, b^-) = \frac{-[23]^3}{[a1][12][3b][b\hat{a}][\hat{a}\hat{b}][\hat{b}a]}.$$

$$\text{Ant}(\hat{a}^-, \hat{b}^- \leftarrow a^-, 1^+, 2^-, 3^+, b^-) = \frac{-[13]^4}{[a1][12][23][3b][b\hat{a}][\hat{a}\hat{b}][\hat{b}a]}.$$

$$\text{Ant}(\hat{a}^-, \hat{b}^- \leftarrow a^+, 1^-, 2^-, 3^+, b^-) = \frac{-[a3]^4}{[a1][12][23][3b][b\hat{a}][\hat{a}\hat{b}][\hat{b}a]}.$$

$$\text{Ant}(\hat{a}^-, \hat{b}^- \leftarrow a^-, 1^+, 2^+, 3^-, b^-) = \frac{-[12]^3}{[a1][23][3b][b\hat{a}][\hat{a}\hat{b}][\hat{b}a]}.$$

$$\text{Ant}(\hat{a}^-, \hat{b}^- \leftarrow a^+, 1^-, 2^+, 3^-, b^-) = \frac{-[a2]^4}{[a1][12][23][3b][b\hat{a}][\hat{a}\hat{b}][\hat{b}a]}.$$

$$\text{Ant}(\hat{a}^-, \hat{b}^- \leftarrow a^+, 1^+, 2^-, 3^-, b^-) = \frac{-[a1]^3}{[12][23][3b][b\hat{a}][\hat{a}\hat{b}][\hat{b}a]}.$$

$$\begin{aligned}
 \text{Ant}(\hat{a}^-, \hat{b}^- \leftarrow a^+, 1^+, 2^-, 3^+, b^+) = & \\
 - \frac{\langle \hat{a}\hat{b} \rangle^3 \langle 2|3+b|\eta \rangle^3}{s_{b23} \langle 23 \rangle \langle 3b \rangle \langle a1 \rangle \langle \hat{a}\hat{b} \rangle \langle b|2+3|\eta \rangle \langle 1|2+3+b|\eta \rangle \langle \hat{a}|2+3+b|\eta \rangle} & \\
 + \frac{\langle \hat{a}\hat{b} \rangle^3 \langle 2|1+3|\eta \rangle^4}{s_{123} \langle 12 \rangle \langle 23 \rangle \langle \hat{a}\hat{b} \rangle \langle \hat{a}\hat{b} \rangle \langle 1|2+3|\eta \rangle \langle 3|1+2|\eta \rangle \langle a|1+2+3|\eta \rangle \langle b|1+2+3|\eta \rangle} & \\
 - \frac{\langle \hat{a}\hat{b} \rangle^3 \langle 2|a+1|\eta \rangle^3}{s_{a12} \langle 12 \rangle \langle 3b \rangle \langle a1 \rangle \langle \hat{a}\hat{b} \rangle \langle a|1+2|\eta \rangle \langle 3|a+1+2|\eta \rangle \langle \hat{b}|a+1+2|\eta \rangle} & \\
 - \frac{\langle \hat{a}\hat{b} \rangle^3 \langle 2|1+3+b|\eta \rangle^4}{s_{b123} \langle 12 \rangle \langle 23 \rangle \langle 3b \rangle \langle \hat{a}\hat{b} \rangle \langle 1|2+3+b|\eta \rangle \langle b|1+2+3|\eta \rangle \langle a|1+2+3+b|\eta \rangle \langle \hat{a}|1+2+3+b|\eta \rangle} & \\
 - \frac{\langle \hat{a}\hat{b} \rangle^3 \langle 2|a+1+3|\eta \rangle^4}{s_{a123} \langle 12 \rangle \langle 23 \rangle \langle a1 \rangle \langle \hat{a}\hat{b} \rangle \langle 3|a+1+2|\eta \rangle \langle a|1+2+3|\eta \rangle \langle b|a+1+2+3|\eta \rangle \langle \hat{b}|a+1+2+3|\eta \rangle} & \\
 + \frac{\langle \hat{a}\hat{b} \rangle^3 [1\eta]^3}{\langle 3b \rangle \langle \hat{a}\hat{b} \rangle \langle \hat{a}\hat{b} \rangle \langle 3|1+2|\eta \rangle \langle a|1+2|\eta \rangle [12][2\eta]} & \\
 + \frac{\langle \hat{a}\hat{b} \rangle^3 [3\eta]^3}{\langle a1 \rangle \langle \hat{a}\hat{b} \rangle \langle \hat{a}\hat{b} \rangle \langle 1|2+3|\eta \rangle \langle b|2+3|\eta \rangle [23][2\eta]} &
 \end{aligned}$$

$$\begin{aligned}
 \text{Ant}(\hat{a}^-, \hat{b}^- \leftarrow a^+, 1^-, 2^+, 3^+, b^+) = & \\
 \frac{\langle \hat{a}\hat{b} \rangle^3 \langle 1|2+3|\eta \rangle^3}{s_{123} \langle 12 \rangle \langle 23 \rangle \langle \hat{a}\hat{b} \rangle \langle \hat{a}\hat{b} \rangle \langle 3|1+2|\eta \rangle \langle a|1+2+3|\eta \rangle \langle b|1+2+3|\eta \rangle} & \\
 - \frac{\langle \hat{a}\hat{b} \rangle^3 \langle 1|a+2|\eta \rangle^4}{s_{a12} \langle 12 \rangle \langle 3b \rangle \langle a1 \rangle \langle \hat{a}\hat{b} \rangle \langle 2|a+1|\eta \rangle \langle a|1+2|\eta \rangle \langle 3|a+1+2|\eta \rangle \langle \hat{b}|a+1+2|\eta \rangle} & \\
 - \frac{\langle \hat{a}\hat{b} \rangle^3 \langle 1|2+3+b|\eta \rangle^3}{s_{b123} \langle 12 \rangle \langle 23 \rangle \langle 3b \rangle \langle \hat{a}\hat{b} \rangle \langle b|1+2+3|\eta \rangle \langle a|1+2+3+b|\eta \rangle \langle \hat{a}|1+2+3+b|\eta \rangle} & \\
 - \frac{\langle \hat{a}\hat{b} \rangle^3 \langle 1|a+2+3|\eta \rangle^4}{s_{a123} \langle 12 \rangle \langle 23 \rangle \langle a1 \rangle \langle \hat{a}\hat{b} \rangle \langle 3|a+1+2|\eta \rangle \langle a|1+2+3|\eta \rangle \langle b|a+1+2+3|\eta \rangle \langle \hat{b}|a+1+2+3|\eta \rangle} & \\
 + \frac{\langle \hat{a}\hat{b} \rangle^3 [2\eta]^3}{\langle 3b \rangle \langle \hat{a}\hat{b} \rangle \langle \hat{a}\hat{b} \rangle \langle 3|1+2|\eta \rangle \langle a|1+2|\eta \rangle [12][1\eta]} - \frac{\langle \hat{a}\hat{b} \rangle^3 [a\eta]^3}{\langle 23 \rangle \langle 3b \rangle \langle \hat{a}\hat{b} \rangle \langle 2|a+1|\eta \rangle \langle \hat{b}|a+1|\eta \rangle [1\eta][a1]} &
 \end{aligned}$$

$$\begin{aligned}
 \text{Ant}(\hat{a}^-, \hat{b}^- \leftarrow a^-, 1^+, 2^+, 3^+, b^+) = & \\
 - \frac{\langle \hat{a}\hat{b} \rangle^3 \langle a|1+2|\eta \rangle^3}{s_{a12} \langle 12 \rangle \langle 3b \rangle \langle a1 \rangle \langle \hat{a}\hat{b} \rangle \langle 2|a+1|\eta \rangle \langle 3|a+1+2|\eta \rangle \langle \hat{b}|a+1+2|\eta \rangle} & \\
 - \frac{\langle \hat{a}\hat{b} \rangle^3 \langle a|1+2+3|\eta \rangle^3}{s_{a123} \langle 12 \rangle \langle 23 \rangle \langle a1 \rangle \langle \hat{a}\hat{b} \rangle \langle 3|a+1+2|\eta \rangle \langle b|a+1+2+3|\eta \rangle \langle \hat{b}|a+1+2+3|\eta \rangle} & \\
 - \frac{\langle \hat{a}\hat{b} \rangle^3 [1\eta]^3}{\langle 23 \rangle \langle 3b \rangle \langle \hat{a}\hat{b} \rangle \langle 2|a+1|\eta \rangle \langle \hat{b}|a+1|\eta \rangle [a1][a\eta]} &
 \end{aligned}$$

$$\text{Ant}(\hat{a}^-, \hat{b}^- \leftarrow a^+, 1^+, 2^+, 3^-, b^+) = -\text{Ant}(\hat{b}^-, \hat{a}^- \leftarrow b^-, 3^+, 2^+, 1^+, a^+)$$

$$\text{Ant}(\hat{a}^-, \hat{b}^- \leftarrow a^+, 1^+, 2^+, 3^+, b^-) = -\text{Ant}(\hat{b}^-, \hat{a}^- \leftarrow b^+, 3^-, 2^+, 1^+, a^+)$$

$$\text{Ant}(\hat{a}^-, \hat{b}^+ \leftarrow a^+, 1^+, 2^+, 3^+, b^+) = 0.$$

$$\text{Ant}(\hat{a}^-, \hat{b}^+ \leftarrow a^-, 1^+, 2^+, 3^+, b^+) = 0.$$

$$\text{Ant}(\hat{a}^-, \hat{b}^+ \leftarrow a^+, 1^-, 2^+, 3^+, b^+) = \frac{\langle \hat{a}1 \rangle^4}{\langle a1 \rangle \langle 12 \rangle \langle 23 \rangle \langle 3b \rangle \langle \hat{b}a \rangle \langle \hat{a}\hat{b} \rangle \langle \hat{b}a \rangle}.$$

$$\text{Ant}(\hat{a}^-, \hat{b}^+ \leftarrow a^+, 1^+, 2^-, 3^+, b^+) = \frac{\langle \hat{a}2 \rangle^4}{\langle a1 \rangle \langle 12 \rangle \langle 23 \rangle \langle 3b \rangle \langle \hat{b}a \rangle \langle \hat{a}\hat{b} \rangle \langle \hat{b}a \rangle}.$$

$$\text{Ant}(\hat{a}^-, \hat{b}^+ \leftarrow a^+, 1^+, 2^+, 3^-, b^+) = \frac{\langle \hat{a}3 \rangle^4}{\langle a1 \rangle \langle 12 \rangle \langle 23 \rangle \langle 3b \rangle \langle \hat{b}a \rangle \langle \hat{a}\hat{b} \rangle \langle \hat{b}a \rangle}.$$

$$\text{Ant}(\hat{a}^-, \hat{b}^+ \leftarrow a^+, 1^+, 2^+, 3^+, b^-) = \frac{\langle \hat{a}b \rangle^4}{\langle a1 \rangle \langle 12 \rangle \langle 23 \rangle \langle 3b \rangle \langle \hat{b}a \rangle \langle \hat{a}\hat{b} \rangle \langle \hat{b}a \rangle}.$$

$$\text{Ant}(\hat{a}^-, \hat{b}^+ \leftarrow a^-, 1^-, 2^-, 3^-, b^+) = 0.$$

$$\text{Ant}(\hat{a}^-, \hat{b}^+ \leftarrow a^-, 1^-, 2^-, 3^+, b^-) = \frac{-[3\hat{b}]^4}{[a1][12][23][3b][\hat{b}a][\hat{a}\hat{b}][\hat{b}a]}.$$

$$\text{Ant}(\hat{a}^-, \hat{b}^+ \leftarrow a^-, 1^-, 2^+, 3^-, b^-) = \frac{-[2\hat{b}]^4}{[a1][12][23][3b][\hat{b}a][\hat{a}\hat{b}][\hat{b}a]}.$$

$$\text{Ant}(\hat{a}^-, \hat{b}^+ \leftarrow a^-, 1^+, 2^-, 3^-, b^-) = \frac{-[1\hat{b}]^4}{[a1][12][23][3b][\hat{b}a][\hat{a}\hat{b}][\hat{b}a]}.$$

$$\text{Ant}(\hat{a}^-, \hat{b}^+ \leftarrow a^+, 1^-, 2^-, 3^-, b^-) = \frac{-[3\hat{b}]^4}{[a1][12][23][3b][\hat{b}a][\hat{a}\hat{b}][\hat{b}a]}.$$

$$\text{Ant}(\hat{a}^-, \hat{b}^+ \leftarrow a^-, 1^-, 2^-, 3^-, b^-) = 0.$$

$$\text{Ant}(\hat{a}^-, \hat{b}^+ \leftarrow a^-, 1^-, 2^+, 3^+, b^+) = 0.$$

$$\begin{aligned} \text{Ant}(\hat{a}^-, \hat{b}^+ \leftarrow a^-, 1^+, 2^-, 3^+, b^+) = & \\ & \frac{\langle a2 \rangle^4 \langle \hat{a}1 + 2|\eta \rangle^4}{s_{a12} \langle 12 \rangle \langle 3b \rangle \langle a1 \rangle \langle \hat{a}b \rangle \langle \hat{a}\hat{b} \rangle \langle 2|a + 1|\eta \rangle \langle a|1 + 2|\eta \rangle \langle 3|a + 1 + 2|\eta \rangle \langle \hat{b}|a + 1 + 2|\eta \rangle} \\ & \frac{\langle a2 \rangle^4 \langle \hat{a}1 + 2 + 3|\eta \rangle^4}{s_{a123} \langle 12 \rangle \langle 23 \rangle \langle a1 \rangle \langle \hat{a}b \rangle \langle \hat{a}\hat{b} \rangle \langle 3|a + 1 + 2|\eta \rangle \langle a|1 + 2 + 3|\eta \rangle \langle b|a + 1 + 2 + 3|\eta \rangle \langle \hat{b}|a + 1 + 2 + 3|\eta \rangle} \\ & \frac{\langle \hat{a}2 \rangle^4 [1\eta]^3}{\langle 23 \rangle \langle 3b \rangle \langle \hat{a}b \rangle \langle \hat{a}\hat{b} \rangle \langle 2|a + 1|\eta \rangle \langle \hat{b}|a + 1|\eta \rangle [a1][a\eta]} \end{aligned}$$

$$\begin{aligned} \text{Ant}(\hat{a}^-, \hat{b}^+ \leftarrow a^-, 1^+, 2^+, 3^-, b^+) = & \\ & \frac{\langle \hat{a}3 \rangle^4 \langle a|1 + 2|\eta \rangle^3}{s_{a12} \langle 12 \rangle \langle 3b \rangle \langle a1 \rangle \langle \hat{a}b \rangle \langle \hat{a}\hat{b} \rangle \langle 2|a + 1|\eta \rangle \langle 3|a + 1 + 2|\eta \rangle \langle \hat{b}|a + 1 + 2|\eta \rangle} \\ & \frac{\langle a3 \rangle^4 \langle \hat{a}1 + 2 + 3|\eta \rangle^4}{s_{a123} \langle 12 \rangle \langle 23 \rangle \langle a1 \rangle \langle \hat{a}b \rangle \langle \hat{a}\hat{b} \rangle \langle 3|a + 1 + 2|\eta \rangle \langle a|1 + 2 + 3|\eta \rangle \langle b|a + 1 + 2 + 3|\eta \rangle \langle \hat{b}|a + 1 + 2 + 3|\eta \rangle} \end{aligned}$$

$$\begin{aligned}
 & - \frac{\langle \hat{a}3 \rangle^4 [1\eta]^3}{\langle 23 \rangle \langle 3b \rangle \langle \hat{a}b \rangle \langle \hat{a}\hat{b} \rangle \langle 2|a+1|\eta \rangle \langle \hat{b}|a+1|\eta \rangle [a1][a\eta]} \\
 \text{Ant}(\hat{a}^-, \hat{b}^+ \leftarrow a^-, 1^+, 2^+, 3^+, b^-) = & \\
 & - \frac{\langle \hat{a}b \rangle^3 \langle a|1+2|\eta \rangle^3}{s_{a12} \langle 12 \rangle \langle 3b \rangle \langle a1 \rangle \langle \hat{a}\hat{b} \rangle \langle 2|a+1|\eta \rangle \langle 3|a+1+2|\eta \rangle \langle \hat{b}|a+1+2|\eta \rangle} \\
 & - \frac{\langle \hat{a}b \rangle^3 \langle a|1+2+3|\eta \rangle^3}{s_{a123} \langle 12 \rangle \langle 23 \rangle \langle a1 \rangle \langle \hat{a}\hat{b} \rangle \langle 3|a+1+2|\eta \rangle \langle b|a+1+2+3|\eta \rangle \langle \hat{b}|a+1+2+3|\eta \rangle} \\
 & - \frac{\langle \hat{a}b \rangle^3 [1\eta]^3}{\langle 23 \rangle \langle 3b \rangle \langle \hat{a}\hat{b} \rangle \langle 2|a+1|\eta \rangle \langle \hat{b}|a+1|\eta \rangle [a1][a\eta]}
 \end{aligned}$$

$$\begin{aligned}
 \text{Ant}(\hat{a}^-, \hat{b}^+ \leftarrow a^+, 1^-, 2^+, 3^+, b^-) = & \\
 & - \frac{\langle \hat{a}1 \rangle^4 \langle b|2+3|\eta \rangle^3}{s_{b23} \langle 23 \rangle \langle 3b \rangle \langle a1 \rangle \langle \hat{a}\hat{b} \rangle \langle 2|3+b|\eta \rangle \langle 1|2+3+b|\eta \rangle \langle \hat{a}|2+3+b|\eta \rangle} \\
 + & \frac{\langle \hat{a}b \rangle^3 \langle 1|2+3|\eta \rangle^3}{s_{123} \langle 12 \rangle \langle 23 \rangle \langle \hat{a}\hat{b} \rangle \langle \hat{a}\hat{b} \rangle \langle 3|1+2|\eta \rangle \langle a|1+2+3|\eta \rangle \langle b|1+2+3|\eta \rangle} \\
 & - \frac{\langle \hat{a}b \rangle^3 \langle 1|a+2|\eta \rangle^4}{s_{a12} \langle 12 \rangle \langle 3b \rangle \langle a1 \rangle \langle \hat{a}\hat{b} \rangle \langle 2|a+1|\eta \rangle \langle a|1+2|\eta \rangle \langle 3|a+1+2|\eta \rangle \langle \hat{b}|a+1+2|\eta \rangle} \\
 & - \frac{\langle 1b \rangle^4 \langle \hat{a}|1+2+3+b|\eta \rangle^3}{s_{b123} \langle 12 \rangle \langle 23 \rangle \langle 3b \rangle \langle \hat{a}\hat{b} \rangle \langle \hat{a}\hat{b} \rangle \langle 1|2+3+b|\eta \rangle \langle b|1+2+3|\eta \rangle \langle a|1+2+3+b|\eta \rangle} \\
 & - \frac{\langle \hat{a}b \rangle^3 \langle 1|a+2+3|\eta \rangle^4}{s_{a123} \langle 12 \rangle \langle 23 \rangle \langle a1 \rangle \langle \hat{a}\hat{b} \rangle \langle 3|a+1+2|\eta \rangle \langle a|1+2+3|\eta \rangle \langle b|a+1+2+3|\eta \rangle \langle \hat{b}|a+1+2+3|\eta \rangle} \\
 + & \frac{\langle \hat{a}b \rangle^3 [2\eta]^3}{\langle 3b \rangle \langle \hat{a}\hat{b} \rangle \langle \hat{a}\hat{b} \rangle \langle 3|1+2|\eta \rangle \langle a|1+2|\eta \rangle [12][1\eta]} \\
 & - \frac{\langle \hat{a}b \rangle^3 [a\eta]^3}{\langle 23 \rangle \langle 3b \rangle \langle \hat{a}\hat{b} \rangle \langle 2|a+1|\eta \rangle \langle \hat{b}|a+1|\eta \rangle [1\eta][a1]} - \frac{\langle \hat{a}1 \rangle^4 [3\eta]^3}{\langle 12 \rangle \langle a1 \rangle \langle \hat{a}\hat{b} \rangle \langle \hat{a}\hat{b} \rangle \langle 2|3+b|\eta \rangle \langle \hat{a}|3+b|\eta \rangle [3b][b\eta]}
 \end{aligned}$$

$$\begin{aligned}
 \text{Ant}(\hat{a}^-, \hat{b}^+ \leftarrow a^+, 1^-, 2^+, 3^-, b^+) = & \\
 & - \frac{\langle \hat{a}1 \rangle^4 \langle 3|2+b|\eta \rangle^4}{s_{b23} \langle 23 \rangle \langle 3b \rangle \langle a1 \rangle \langle \hat{a}\hat{b} \rangle \langle \hat{a}\hat{b} \rangle \langle 2|3+b|\eta \rangle \langle b|2+3|\eta \rangle \langle 1|2+3+b|\eta \rangle \langle \hat{a}|2+3+b|\eta \rangle} \\
 + & \frac{\langle 13 \rangle^4 \langle \hat{a}|1+2+3|\eta \rangle^4}{s_{123} \langle 12 \rangle \langle 23 \rangle \langle \hat{a}\hat{b} \rangle \langle \hat{a}\hat{b} \rangle \langle \hat{a}\hat{b} \rangle \langle 1|2+3|\eta \rangle \langle 3|1+2|\eta \rangle \langle a|1+2+3|\eta \rangle \langle b|1+2+3|\eta \rangle} \\
 & - \frac{\langle \hat{a}3 \rangle^4 \langle 1|a+2|\eta \rangle^4}{s_{a12} \langle 12 \rangle \langle 3b \rangle \langle a1 \rangle \langle \hat{a}\hat{b} \rangle \langle \hat{a}\hat{b} \rangle \langle 2|a+1|\eta \rangle \langle a|1+2|\eta \rangle \langle 3|a+1+2|\eta \rangle \langle \hat{b}|a+1+2|\eta \rangle} \\
 & - \frac{\langle 13 \rangle^4 \langle \hat{a}|1+2+3+b|\eta \rangle^3}{s_{b123} \langle 12 \rangle \langle 23 \rangle \langle 3b \rangle \langle \hat{a}\hat{b} \rangle \langle \hat{a}\hat{b} \rangle \langle 1|2+3+b|\eta \rangle \langle b|1+2+3|\eta \rangle \langle a|1+2+3+b|\eta \rangle} \\
 & - \frac{\langle 13 \rangle^4 \langle \hat{a}|1+2+3|\eta \rangle^4}{s_{a123} \langle 12 \rangle \langle 23 \rangle \langle a1 \rangle \langle \hat{a}\hat{b} \rangle \langle \hat{a}\hat{b} \rangle \langle 3|a+1+2|\eta \rangle \langle a|1+2+3|\eta \rangle \langle b|a+1+2+3|\eta \rangle \langle \hat{b}|a+1+2+3|\eta \rangle} \\
 + & \frac{\langle \hat{a}3 \rangle^4 [2\eta]^3}{\langle 3b \rangle \langle \hat{a}\hat{b} \rangle \langle \hat{a}\hat{b} \rangle \langle \hat{a}\hat{b} \rangle \langle 3|1+2|\eta \rangle \langle a|1+2|\eta \rangle [12][1\eta]}
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{\langle \hat{a}1 \rangle^4 [2\eta]^3}{\langle a1 \rangle \langle \hat{a}b \rangle \langle \hat{a}b \rangle \langle \hat{a}b \rangle \langle 1|2+3|\eta \rangle \langle b|2+3|\eta \rangle [23][3\eta]} \\
 & - \frac{\langle \hat{a}3 \rangle^4 [a\eta]^3}{\langle 23 \rangle \langle 3b \rangle \langle \hat{a}b \rangle \langle \hat{a}b \rangle \langle 2|a+1|\eta \rangle \langle \hat{b}|a+1|\eta \rangle [1\eta][a1]} \\
 & - \frac{\langle \hat{a}1 \rangle^4 [b\eta]^3}{\langle 12 \rangle \langle a1 \rangle \langle \hat{a}b \rangle \langle \hat{a}b \rangle \langle 2|3+b|\eta \rangle \langle \hat{a}|3+b|\eta \rangle [3b][3\eta]}
 \end{aligned}$$

$$\text{Ant}(\hat{a}^-, \hat{b}^+ \leftarrow a^+, 1^-, 2^-, 3^+, b^+) =$$

$$\begin{aligned}
 & - \frac{\langle \hat{a}1 \rangle^4 \langle 2|3+b|\eta \rangle^3}{s_{b23} \langle 23 \rangle \langle 3b \rangle \langle a1 \rangle \langle \hat{a}b \rangle \langle \hat{a}b \rangle \langle b|2+3|\eta \rangle \langle 1|2+3+b|\eta \rangle \langle \hat{a}|2+3+b|\eta \rangle} \\
 & + \frac{\langle 12 \rangle^3 \langle \hat{a}|1+2+3|\eta \rangle^4}{s_{123} \langle 23 \rangle \langle \hat{a}b \rangle \langle \hat{a}b \rangle \langle \hat{a}b \rangle \langle 1|2+3|\eta \rangle \langle 3|1+2|\eta \rangle \langle a|1+2+3|\eta \rangle \langle b|1+2+3|\eta \rangle} \\
 & - \frac{\langle 12 \rangle^3 \langle \hat{a}|1+2|\eta \rangle^4}{s_{a12} \langle 3b \rangle \langle a1 \rangle \langle \hat{a}b \rangle \langle \hat{a}b \rangle \langle 2|a+1|\eta \rangle \langle a|1+2|\eta \rangle \langle 3|a+1+2|\eta \rangle \langle \hat{b}|a+1+2|\eta \rangle} \\
 & - \frac{\langle 12 \rangle^3 \langle \hat{a}|1+2+3+b|\eta \rangle^3}{s_{b123} \langle 23 \rangle \langle 3b \rangle \langle \hat{a}b \rangle \langle \hat{a}b \rangle \langle 1|2+3+b|\eta \rangle \langle b|1+2+3|\eta \rangle \langle a|1+2+3+b|\eta \rangle} \\
 & - \frac{\langle 12 \rangle^3 \langle \hat{a}|1+2+3|\eta \rangle^4}{s_{a123} \langle 23 \rangle \langle a1 \rangle \langle \hat{a}b \rangle \langle \hat{a}b \rangle \langle 3|a+1+2|\eta \rangle \langle a|1+2+3|\eta \rangle \langle b|a+1+2+3|\eta \rangle \langle \hat{b}|a+1+2+3|\eta \rangle} \\
 & + \frac{\langle \hat{a}|1+2|\eta \rangle^4}{\langle 3b \rangle \langle \hat{a}b \rangle \langle \hat{a}b \rangle \langle \hat{a}b \rangle \langle 3|1+2|\eta \rangle \langle a|1+2|\eta \rangle [12][1\eta][2\eta]} \\
 & + \frac{\langle \hat{a}1 \rangle^4 [3\eta]^3}{\langle a1 \rangle \langle \hat{a}b \rangle \langle \hat{a}b \rangle \langle \hat{a}b \rangle \langle 1|2+3|\eta \rangle \langle b|2+3|\eta \rangle [23][2\eta]} \\
 & - \frac{\langle \hat{a}2 \rangle^4 [a\eta]^3}{\langle 23 \rangle \langle 3b \rangle \langle \hat{a}b \rangle \langle \hat{a}b \rangle \langle 2|a+1|\eta \rangle \langle \hat{b}|a+1|\eta \rangle [1\eta][a1]}
 \end{aligned}$$

$$\text{Ant}(\hat{a}^-, \hat{b}^+ \leftarrow a^+, 1^+, 2^-, 3^-, b^+) =$$

$$\begin{aligned}
 & - \frac{\langle 23 \rangle^3 \langle \hat{a}|2+3+b|\eta \rangle^3}{s_{b23} \langle 3b \rangle \langle a1 \rangle \langle \hat{a}b \rangle \langle \hat{a}b \rangle \langle 2|3+b|\eta \rangle \langle b|2+3|\eta \rangle \langle 1|2+3+b|\eta \rangle} \\
 & + \frac{\langle 23 \rangle^3 \langle \hat{a}|1+2+3|\eta \rangle^4}{s_{123} \langle 12 \rangle \langle \hat{a}b \rangle \langle \hat{a}b \rangle \langle \hat{a}b \rangle \langle 1|2+3|\eta \rangle \langle 3|1+2|\eta \rangle \langle a|1+2+3|\eta \rangle \langle b|1+2+3|\eta \rangle} \\
 & - \frac{\langle \hat{a}3 \rangle^4 \langle 2|a+1|\eta \rangle^3}{s_{a12} \langle 12 \rangle \langle 3b \rangle \langle a1 \rangle \langle \hat{a}b \rangle \langle \hat{a}b \rangle \langle a|1+2|\eta \rangle \langle 3|a+1+2|\eta \rangle \langle \hat{b}|a+1+2|\eta \rangle} \\
 & - \frac{\langle 23 \rangle^3 \langle \hat{a}|1+2+3+b|\eta \rangle^3}{s_{b123} \langle 12 \rangle \langle 3b \rangle \langle \hat{a}b \rangle \langle \hat{a}b \rangle \langle 1|2+3+b|\eta \rangle \langle b|1+2+3|\eta \rangle \langle a|1+2+3+b|\eta \rangle} \\
 & - \frac{\langle 23 \rangle^3 \langle \hat{a}|1+2+3|\eta \rangle^4}{s_{a123} \langle 12 \rangle \langle a1 \rangle \langle \hat{a}b \rangle \langle \hat{a}b \rangle \langle 3|a+1+2|\eta \rangle \langle a|1+2+3|\eta \rangle \langle b|a+1+2+3|\eta \rangle \langle \hat{b}|a+1+2+3|\eta \rangle} \\
 & + \frac{\langle \hat{a}3 \rangle^4 [1\eta]^3}{\langle 3b \rangle \langle \hat{a}b \rangle \langle \hat{a}b \rangle \langle \hat{a}b \rangle \langle 3|1+2|\eta \rangle \langle a|1+2|\eta \rangle [12][2\eta]} \\
 & + \frac{\langle \hat{a}|2+3|\eta \rangle^4}{\langle a1 \rangle \langle \hat{a}b \rangle \langle \hat{a}b \rangle \langle \hat{a}b \rangle \langle 1|2+3|\eta \rangle \langle b|2+3|\eta \rangle [23][2\eta][3\eta]}
 \end{aligned}$$

$$- \frac{\langle \hat{a}2 \rangle^4 [b\eta]^3}{\langle 12 \rangle \langle a1 \rangle \langle \hat{a}\hat{b} \rangle \langle 2|3+b|\eta \rangle \langle \hat{a}|3+b|\eta \rangle [3b][3\eta]}$$

$$\text{Ant}(\hat{a}^-, \hat{b}^+ \leftarrow a^+, 1^+, 2^-, 3^+, b^-) =$$

$$- \frac{\langle 2b \rangle^4 \langle \hat{a}|2+3+b|\eta \rangle^3}{s_{b23} \langle 23 \rangle \langle 3b \rangle \langle a1 \rangle \langle \hat{a}\hat{b} \rangle \langle 2|3+b|\eta \rangle \langle b|2+3|\eta \rangle \langle 1|2+3+b|\eta \rangle}$$

$$+ \frac{\langle \hat{a}\hat{b} \rangle^3 \langle 2|1+3|\eta \rangle^4}{s_{123} \langle 12 \rangle \langle 23 \rangle \langle \hat{a}\hat{b} \rangle \langle 1|2+3|\eta \rangle \langle 3|1+2|\eta \rangle \langle a|1+2+3|\eta \rangle \langle b|1+2+3|\eta \rangle}$$

$$- \frac{\langle \hat{a}\hat{b} \rangle^3 \langle 2|a+1|\eta \rangle^3}{s_{a12} \langle 12 \rangle \langle 3b \rangle \langle a1 \rangle \langle \hat{a}\hat{b} \rangle \langle a|1+2|\eta \rangle \langle 3|a+1+2|\eta \rangle \langle \hat{b}|a+1+2|\eta \rangle}$$

$$- \frac{\langle 2b \rangle^4 \langle \hat{a}|1+2+3+b|\eta \rangle^3}{s_{b123} \langle 12 \rangle \langle 23 \rangle \langle 3b \rangle \langle \hat{a}\hat{b} \rangle \langle 1|2+3+b|\eta \rangle \langle b|1+2+3|\eta \rangle \langle a|1+2+3+b|\eta \rangle}$$

$$- \frac{\langle \hat{a}\hat{b} \rangle^3 \langle 2|a+1+3|\eta \rangle^4}{s_{a123} \langle 12 \rangle \langle 23 \rangle \langle a1 \rangle \langle \hat{a}\hat{b} \rangle \langle 3|a+1+2|\eta \rangle \langle a|1+2+3|\eta \rangle \langle b|a+1+2+3|\eta \rangle \langle \hat{b}|a+1+2+3|\eta \rangle}$$

$$+ \frac{\langle \hat{a}\hat{b} \rangle^3 [1\eta]^3}{\langle 3b \rangle \langle \hat{a}\hat{b} \rangle \langle 3|1+2|\eta \rangle \langle a|1+2|\eta \rangle [12][2\eta]}$$

$$+ \frac{\langle \hat{a}\hat{b} \rangle^3 [3\eta]^3}{\langle a1 \rangle \langle \hat{a}\hat{b} \rangle \langle \hat{a}\hat{b} \rangle \langle 1|2+3|\eta \rangle \langle b|2+3|\eta \rangle [23][2\eta]}$$

$$- \frac{\langle \hat{a}2 \rangle^4 [3\eta]^3}{\langle 12 \rangle \langle a1 \rangle \langle \hat{a}\hat{b} \rangle \langle 2|3+b|\eta \rangle \langle \hat{a}|3+b|\eta \rangle [3b][b\eta]}$$

$$\text{Ant}(\hat{a}^-, \hat{b}^+ \leftarrow a^+, 1^+, 2^+, 3^-, b^-) =$$

$$- \frac{\langle 3b \rangle^3 \langle \hat{a}|2+3+b|\eta \rangle^3}{s_{b23} \langle 23 \rangle \langle a1 \rangle \langle \hat{a}\hat{b} \rangle \langle 2|3+b|\eta \rangle \langle b|2+3|\eta \rangle \langle 1|2+3+b|\eta \rangle}$$

$$+ \frac{\langle \hat{a}\hat{b} \rangle^3 \langle 3|1+2|\eta \rangle^3}{s_{123} \langle 12 \rangle \langle 23 \rangle \langle \hat{a}\hat{b} \rangle \langle 1|2+3|\eta \rangle \langle a|1+2+3|\eta \rangle \langle b|1+2+3|\eta \rangle}$$

$$- \frac{\langle 3b \rangle^3 \langle \hat{a}|1+2+3+b|\eta \rangle^3}{s_{b123} \langle 12 \rangle \langle 23 \rangle \langle \hat{a}\hat{b} \rangle \langle 1|2+3+b|\eta \rangle \langle b|1+2+3|\eta \rangle \langle a|1+2+3+b|\eta \rangle}$$

$$- \frac{\langle \hat{a}\hat{b} \rangle^3 \langle 3|a+1+2|\eta \rangle^3}{s_{a123} \langle 12 \rangle \langle 23 \rangle \langle a1 \rangle \langle \hat{a}\hat{b} \rangle \langle a|1+2+3|\eta \rangle \langle b|a+1+2+3|\eta \rangle \langle \hat{b}|a+1+2+3|\eta \rangle}$$

$$+ \frac{\langle \hat{a}\hat{b} \rangle^3 [2\eta]^3}{\langle a1 \rangle \langle \hat{a}\hat{b} \rangle \langle \hat{a}\hat{b} \rangle \langle 1|2+3|\eta \rangle \langle b|2+3|\eta \rangle [23][3\eta]}$$

$$- \frac{\langle \hat{a}|3+b|\eta \rangle^3}{\langle 12 \rangle \langle a1 \rangle \langle \hat{a}\hat{b} \rangle \langle 2|3+b|\eta \rangle [3b][3\eta][b\eta]}$$

$$\text{Ant}(\hat{a}^+, \hat{b}^+ \leftarrow a^+, 1^+, 2^+, 3^+, b^+) = 0.$$

$$\text{Ant}(\hat{a}^+, \hat{b}^+ \leftarrow a^-, 1^+, 2^+, 3^+, b^+) = 0.$$

$$\text{Ant}(\hat{a}^+, \hat{b}^+ \leftarrow a^+, 1^-, 2^+, 3^+, b^+) = 0.$$

$$\text{Ant}(\hat{a}^+, \hat{b}^+ \leftarrow a^+, 1^+, 2^-, 3^+, b^+) = 0.$$

$$\text{Ant}(\hat{a}^+, \hat{b}^+ \leftarrow a^+, 1^+, 2^+, 3^-, b^+) = 0.$$

$$\text{Ant}(\hat{a}^+, \hat{b}^+ \leftarrow a^+, 1^+, 2^+, 3^+, b^-) = 0.$$

$$\text{Ant}(\hat{a}^+, \hat{b}^+ \leftarrow a^-, 1^-, 2^+, 3^+, b^+) = \frac{\langle a1 \rangle^3}{\langle 12 \rangle \langle 23 \rangle \langle 3b \rangle \langle b\hat{a} \rangle \langle \hat{a}\hat{b} \rangle \langle \hat{b}a \rangle}.$$

$$\text{Ant}(\hat{a}^+, \hat{b}^+ \leftarrow a^-, 1^+, 2^-, 3^+, b^+) = \frac{\langle a2 \rangle^4}{\langle a1 \rangle \langle 12 \rangle \langle 23 \rangle \langle 3b \rangle \langle b\hat{a} \rangle \langle \hat{a}\hat{b} \rangle \langle \hat{b}a \rangle}.$$

$$\text{Ant}(\hat{a}^+, \hat{b}^+ \leftarrow a^-, 1^+, 2^+, 3^-, b^+) = \frac{\langle a3 \rangle^4}{\langle a1 \rangle \langle 12 \rangle \langle 23 \rangle \langle 3b \rangle \langle b\hat{a} \rangle \langle \hat{a}\hat{b} \rangle \langle \hat{b}a \rangle}.$$

$$\text{Ant}(\hat{a}^+, \hat{b}^+ \leftarrow a^-, 1^+, 2^+, 3^+, b^-) = \frac{\langle ab \rangle^4}{\langle a1 \rangle \langle 12 \rangle \langle 23 \rangle \langle 3b \rangle \langle b\hat{a} \rangle \langle \hat{a}\hat{b} \rangle \langle \hat{b}a \rangle}.$$

$$\text{Ant}(\hat{a}^+, \hat{b}^+ \leftarrow a^+, 1^-, 2^-, 3^+, b^+) = \frac{\langle 12 \rangle^3}{\langle a1 \rangle \langle 23 \rangle \langle 3b \rangle \langle b\hat{a} \rangle \langle \hat{a}\hat{b} \rangle \langle \hat{b}a \rangle}.$$

$$\text{Ant}(\hat{a}^+, \hat{b}^+ \leftarrow a^+, 1^-, 2^+, 3^-, b^+) = \frac{\langle 13 \rangle^4}{\langle a1 \rangle \langle 12 \rangle \langle 23 \rangle \langle 3b \rangle \langle b\hat{a} \rangle \langle \hat{a}\hat{b} \rangle \langle \hat{b}a \rangle}.$$

$$\text{Ant}(\hat{a}^+, \hat{b}^+ \leftarrow a^+, 1^-, 2^+, 3^+, b^-) = \frac{\langle 1b \rangle^4}{\langle a1 \rangle \langle 12 \rangle \langle 23 \rangle \langle 3b \rangle \langle b\hat{a} \rangle \langle \hat{a}\hat{b} \rangle \langle \hat{b}a \rangle}.$$

$$\text{Ant}(\hat{a}^+, \hat{b}^+ \leftarrow a^+, 1^+, 2^-, 3^-, b^+) = \frac{\langle 23 \rangle^3}{\langle a1 \rangle \langle 12 \rangle \langle 3b \rangle \langle b\hat{a} \rangle \langle \hat{a}\hat{b} \rangle \langle \hat{b}a \rangle}.$$

$$\text{Ant}(\hat{a}^+, \hat{b}^+ \leftarrow a^+, 1^+, 2^-, 3^+, b^-) = \frac{\langle 2b \rangle^4}{\langle a1 \rangle \langle 12 \rangle \langle 23 \rangle \langle 3b \rangle \langle b\hat{a} \rangle \langle \hat{a}\hat{b} \rangle \langle \hat{b}a \rangle}.$$

$$\text{Ant}(\hat{a}^+, \hat{b}^+ \leftarrow a^+, 1^+, 2^+, 3^-, b^-) = \frac{\langle 3b \rangle^4}{\langle a1 \rangle \langle 12 \rangle \langle 23 \rangle \langle 3b \rangle \langle b\hat{a} \rangle \langle \hat{a}\hat{b} \rangle \langle \hat{b}a \rangle}.$$

$$\text{Ant}(\hat{a}^+, \hat{b}^+ \leftarrow a^-, 1^-, 2^-, 3^-, b^-) = \frac{-[\hat{a}\hat{b}]^3}{[a1][12][23][3b][b\hat{a}][\hat{b}a]}.$$

$$\text{Ant}(\hat{a}^+, \hat{b}^+ \leftarrow a^-, 1^-, 2^-, 3^+, b^+) =$$

$$\begin{aligned} & - \frac{\langle a1 \rangle^3 \langle 2|3 + b|\eta \rangle^3}{s_{b23} \langle 23 \rangle \langle 3b \rangle \langle a\hat{b} \rangle \langle \hat{a}\hat{b} \rangle \langle b|2 + 3|\eta \rangle \langle 1|2 + 3 + b|\eta \rangle \langle \hat{a}|2 + 3 + b|\eta \rangle} \\ & + \frac{\langle 12 \rangle^3 \langle a|1 + 2 + 3|\eta \rangle^3}{s_{123} \langle 23 \rangle \langle a\hat{b} \rangle \langle \hat{a}\hat{b} \rangle \langle \hat{a}\hat{b} \rangle \langle 1|2 + 3|\eta \rangle \langle 3|1 + 2|\eta \rangle \langle b|1 + 2 + 3|\eta \rangle} \\ & - \frac{\langle 12 \rangle^3 \langle a|1 + 2 + 3 + b|\eta \rangle^3}{s_{b123} \langle 23 \rangle \langle 3b \rangle \langle a\hat{b} \rangle \langle \hat{a}\hat{b} \rangle \langle 1|2 + 3 + b|\eta \rangle \langle b|1 + 2 + 3|\eta \rangle \langle \hat{a}|1 + 2 + 3 + b|\eta \rangle} \\ & + \frac{\langle a|1 + 2|\eta \rangle^3}{\langle 3b \rangle \langle a\hat{b} \rangle \langle \hat{a}\hat{b} \rangle \langle \hat{a}\hat{b} \rangle \langle 3|1 + 2|\eta \rangle [12][1\eta][2\eta]} \\ & + \frac{\langle a1 \rangle^3 [3\eta]^3}{\langle a\hat{b} \rangle \langle \hat{a}\hat{b} \rangle \langle \hat{a}\hat{b} \rangle \langle 1|2 + 3|\eta \rangle \langle b|2 + 3|\eta \rangle [23][2\eta]} \\ & - \frac{\langle 2|a + 1|\eta \rangle^3}{\langle 23 \rangle \langle 3b \rangle \langle a\hat{b} \rangle \langle \hat{a}\hat{b} \rangle \langle \hat{b}|a + 1|\eta \rangle [1\eta][a1][a\eta]} \end{aligned}$$

$$\text{Ant}(\hat{a}^+, \hat{b}^+ \leftarrow a^-, 1^-, 2^+, 3^-, b^+) =$$

$$\begin{aligned}
 & - \frac{\langle a1 \rangle^3 \langle 3|2+b|\eta \rangle^4}{s_{b23} \langle 23 \rangle \langle 3b \rangle \langle \hat{a}\hat{b} \rangle \langle \hat{a}\hat{b} \rangle \langle 2|3+b|\eta \rangle \langle b|2+3|\eta \rangle \langle 1|2+3+b|\eta \rangle \langle \hat{a}|2+3+b|\eta \rangle} \\
 & + \frac{\langle 13 \rangle^4 \langle a|1+2+3|\eta \rangle^3}{s_{123} \langle 12 \rangle \langle 23 \rangle \langle \hat{a}\hat{b} \rangle \langle \hat{a}\hat{b} \rangle \langle \hat{a}\hat{b} \rangle \langle 1|2+3|\eta \rangle \langle 3|1+2|\eta \rangle \langle b|1+2+3|\eta \rangle} \\
 & - \frac{\langle a1 \rangle^3 \langle 3|a+1+2|\eta \rangle^3}{s_{a12} \langle 12 \rangle \langle 3b \rangle \langle \hat{a}\hat{b} \rangle \langle \hat{a}\hat{b} \rangle \langle 2|a+1|\eta \rangle \langle a|1+2|\eta \rangle \langle \hat{b}|a+1+2|\eta \rangle} \\
 & - \frac{\langle 13 \rangle^4 \langle a|1+2+3+b|\eta \rangle^3}{s_{b123} \langle 12 \rangle \langle 23 \rangle \langle 3b \rangle \langle \hat{a}\hat{b} \rangle \langle \hat{a}\hat{b} \rangle \langle 1|2+3+b|\eta \rangle \langle b|1+2+3|\eta \rangle \langle \hat{a}|1+2+3+b|\eta \rangle} \\
 & + \frac{\langle a3 \rangle^4 [2\eta]^3}{\langle 3b \rangle \langle \hat{a}\hat{b} \rangle \langle \hat{a}\hat{b} \rangle \langle \hat{a}\hat{b} \rangle \langle 3|1+2|\eta \rangle \langle a|1+2|\eta \rangle [12][1\eta]} \\
 & + \frac{\langle a1 \rangle^3 [2\eta]^3}{\langle \hat{a}\hat{b} \rangle \langle \hat{a}\hat{b} \rangle \langle \hat{a}\hat{b} \rangle \langle 1|2+3|\eta \rangle \langle b|2+3|\eta \rangle [23][3\eta]} \\
 & - \frac{\langle 3|a+1|\eta \rangle^4}{\langle 23 \rangle \langle 3b \rangle \langle \hat{a}\hat{b} \rangle \langle \hat{a}\hat{b} \rangle \langle 2|a+1|\eta \rangle \langle \hat{b}|a+1|\eta \rangle [1\eta][a1][a\eta]} \\
 & - \frac{\langle a1 \rangle^3 [b\eta]^3}{\langle 12 \rangle \langle \hat{a}\hat{b} \rangle \langle \hat{a}\hat{b} \rangle \langle 2|3+b|\eta \rangle \langle \hat{a}|3+b|\eta \rangle [3b][3\eta]}
 \end{aligned}$$

$$\text{Ant}(\hat{a}^+, \hat{b}^+ \leftarrow a^-, 1^-, 2^+, 3^+, b^-) =$$

$$\begin{aligned}
 & - \frac{\langle a1 \rangle^3 \langle b|2+3|\eta \rangle^3}{s_{b23} \langle 23 \rangle \langle 3b \rangle \langle \hat{a}\hat{b} \rangle \langle \hat{a}\hat{b} \rangle \langle 2|3+b|\eta \rangle \langle 1|2+3+b|\eta \rangle \langle \hat{a}|2+3+b|\eta \rangle} \\
 & + \frac{\langle ab \rangle^4 \langle 1|2+3|\eta \rangle^3}{s_{123} \langle 12 \rangle \langle 23 \rangle \langle \hat{a}\hat{b} \rangle \langle \hat{a}\hat{b} \rangle \langle \hat{a}\hat{b} \rangle \langle 3|1+2|\eta \rangle \langle a|1+2+3|\eta \rangle \langle b|1+2+3|\eta \rangle} \\
 & - \frac{\langle a1 \rangle^3 \langle b|a+1+2|\eta \rangle^4}{s_{a12} \langle 12 \rangle \langle 3b \rangle \langle \hat{a}\hat{b} \rangle \langle \hat{a}\hat{b} \rangle \langle 2|a+1|\eta \rangle \langle a|1+2|\eta \rangle \langle 3|a+1+2|\eta \rangle \langle \hat{b}|a+1+2|\eta \rangle} \\
 & - \frac{\langle 1b \rangle^4 \langle a|1+2+3+b|\eta \rangle^3}{s_{b123} \langle 12 \rangle \langle 23 \rangle \langle 3b \rangle \langle \hat{a}\hat{b} \rangle \langle \hat{a}\hat{b} \rangle \langle 1|2+3+b|\eta \rangle \langle b|1+2+3|\eta \rangle \langle \hat{a}|1+2+3+b|\eta \rangle} \\
 & - \frac{\langle a1 \rangle^3 \langle b|a+1+2+3|\eta \rangle^3}{s_{a123} \langle 12 \rangle \langle 23 \rangle \langle \hat{a}\hat{b} \rangle \langle \hat{a}\hat{b} \rangle \langle 3|a+1+2|\eta \rangle \langle a|1+2+3|\eta \rangle \langle \hat{b}|a+1+2+3|\eta \rangle} \\
 & + \frac{\langle ab \rangle^4 [2\eta]^3}{\langle 3b \rangle \langle \hat{a}\hat{b} \rangle \langle \hat{a}\hat{b} \rangle \langle \hat{a}\hat{b} \rangle \langle 3|1+2|\eta \rangle \langle a|1+2|\eta \rangle [12][1\eta]} \\
 & - \frac{\langle b|a+1|\eta \rangle^4}{\langle 23 \rangle \langle 3b \rangle \langle \hat{a}\hat{b} \rangle \langle \hat{a}\hat{b} \rangle \langle 2|a+1|\eta \rangle \langle \hat{b}|a+1|\eta \rangle [1\eta][a1][a\eta]} \\
 & - \frac{\langle a1 \rangle^3 [3\eta]^3}{\langle 12 \rangle \langle \hat{a}\hat{b} \rangle \langle \hat{a}\hat{b} \rangle \langle 2|3+b|\eta \rangle \langle \hat{a}|3+b|\eta \rangle [3b][b\eta]}
 \end{aligned}$$

$$\text{Ant}(\hat{a}^+, \hat{b}^+ \leftarrow a^-, 1^+, 2^-, 3^-, b^+) =$$

$$\begin{aligned}
 & - \frac{\langle 23 \rangle^3 \langle a|2+3+b|\eta \rangle^4}{s_{b23} \langle 3b \rangle \langle a1 \rangle \langle \hat{a}\hat{b} \rangle \langle \hat{a}\hat{b} \rangle \langle 2|3+b|\eta \rangle \langle b|2+3|\eta \rangle \langle 1|2+3+b|\eta \rangle \langle \hat{a}|2+3+b|\eta \rangle} \\
 & + \frac{\langle 23 \rangle^3 \langle a|1+2+3|\eta \rangle^3}{s_{123} \langle 12 \rangle \langle \hat{a}\hat{b} \rangle \langle \hat{a}\hat{b} \rangle \langle \hat{a}\hat{b} \rangle \langle 1|2+3|\eta \rangle \langle 3|1+2|\eta \rangle \langle b|1+2+3|\eta \rangle}
 \end{aligned}$$

$$\begin{aligned}
 & - \frac{\langle a2 \rangle^4 \langle 3|a+1+2|\eta \rangle^3}{s_{a12} \langle 12 \rangle \langle 3b \rangle \langle a1 \rangle \langle \hat{a}b \rangle \langle \hat{a}\hat{b} \rangle \langle 2|a+1|\eta \rangle \langle a|1+2|\eta \rangle \langle \hat{b}|a+1+2|\eta \rangle} \\
 & - \frac{\langle 23 \rangle^3 \langle a|1+2+3+b|\eta \rangle^3}{s_{b123} \langle 12 \rangle \langle 3b \rangle \langle \hat{a}\hat{b} \rangle \langle \hat{a}\hat{b} \rangle \langle 1|2+3+b|\eta \rangle \langle b|1+2+3|\eta \rangle \langle \hat{a}|1+2+3+b|\eta \rangle} \\
 & + \frac{\langle a3 \rangle^4 [1\eta]^3}{\langle 3b \rangle \langle \hat{a}\hat{b} \rangle \langle \hat{a}\hat{b} \rangle \langle \hat{a}\hat{b} \rangle \langle 3|1+2|\eta \rangle \langle a|1+2|\eta \rangle [12][2\eta]} \\
 & + \frac{\langle a|2+3|\eta \rangle^4}{\langle a1 \rangle \langle \hat{a}\hat{b} \rangle \langle \hat{a}\hat{b} \rangle \langle \hat{a}\hat{b} \rangle \langle 1|2+3|\eta \rangle \langle b|2+3|\eta \rangle [23][2\eta][3\eta]} \\
 & - \frac{\langle 23 \rangle^3 [1\eta]^3}{\langle 3b \rangle \langle \hat{a}\hat{b} \rangle \langle \hat{a}\hat{b} \rangle \langle 2|a+1|\eta \rangle \langle \hat{b}|a+1|\eta \rangle [a1][a\eta]} \\
 & - \frac{\langle a2 \rangle^4 [b\eta]^3}{\langle 12 \rangle \langle a1 \rangle \langle \hat{a}\hat{b} \rangle \langle \hat{a}\hat{b} \rangle \langle 2|3+b|\eta \rangle \langle \hat{a}|3+b|\eta \rangle [3b][3\eta]}
 \end{aligned}$$

$$\text{Ant}(\hat{a}^+, \hat{b}^+ \leftarrow a^-, 1^+, 2^-, 3^+, b^-) =$$

$$\begin{aligned}
 & - \frac{\langle 2b \rangle^4 \langle a|2+3+b|\eta \rangle^4}{s_{b23} \langle 23 \rangle \langle 3b \rangle \langle a1 \rangle \langle \hat{a}\hat{b} \rangle \langle \hat{a}\hat{b} \rangle \langle 2|3+b|\eta \rangle \langle b|2+3|\eta \rangle \langle 1|2+3+b|\eta \rangle \langle \hat{a}|2+3+b|\eta \rangle} \\
 & + \frac{\langle ab \rangle^4 \langle 2|1+3|\eta \rangle^4}{s_{123} \langle 12 \rangle \langle 23 \rangle \langle \hat{a}\hat{b} \rangle \langle \hat{a}\hat{b} \rangle \langle \hat{a}\hat{b} \rangle \langle 1|2+3|\eta \rangle \langle 3|1+2|\eta \rangle \langle a|1+2+3|\eta \rangle \langle b|1+2+3|\eta \rangle} \\
 & - \frac{\langle a2 \rangle^4 \langle b|a+1+2|\eta \rangle^4}{s_{a12} \langle 12 \rangle \langle 3b \rangle \langle a1 \rangle \langle \hat{a}\hat{b} \rangle \langle \hat{a}\hat{b} \rangle \langle 2|a+1|\eta \rangle \langle a|1+2|\eta \rangle \langle 3|a+1+2|\eta \rangle \langle \hat{b}|a+1+2|\eta \rangle} \\
 & - \frac{\langle 2b \rangle^4 \langle a|1+2+3+b|\eta \rangle^3}{s_{b123} \langle 12 \rangle \langle 23 \rangle \langle 3b \rangle \langle \hat{a}\hat{b} \rangle \langle \hat{a}\hat{b} \rangle \langle 1|2+3+b|\eta \rangle \langle b|1+2+3|\eta \rangle \langle \hat{a}|1+2+3+b|\eta \rangle} \\
 & - \frac{\langle a2 \rangle^4 \langle b|a+1+2+3|\eta \rangle^3}{s_{a123} \langle 12 \rangle \langle 23 \rangle \langle a1 \rangle \langle \hat{a}\hat{b} \rangle \langle \hat{a}\hat{b} \rangle \langle 3|a+1+2|\eta \rangle \langle a|1+2+3|\eta \rangle \langle \hat{b}|a+1+2+3|\eta \rangle} \\
 & + \frac{\langle ab \rangle^4 [1\eta]^3}{\langle 3b \rangle \langle \hat{a}\hat{b} \rangle \langle \hat{a}\hat{b} \rangle \langle \hat{a}\hat{b} \rangle \langle 3|1+2|\eta \rangle \langle a|1+2|\eta \rangle [12][2\eta]} \\
 & + \frac{\langle ab \rangle^4 [3\eta]^3}{\langle a1 \rangle \langle \hat{a}\hat{b} \rangle \langle \hat{a}\hat{b} \rangle \langle \hat{a}\hat{b} \rangle \langle 1|2+3|\eta \rangle \langle b|2+3|\eta \rangle [23][2\eta]} \\
 & - \frac{\langle 2b \rangle^4 [1\eta]^3}{\langle 23 \rangle \langle 3b \rangle \langle \hat{a}\hat{b} \rangle \langle \hat{a}\hat{b} \rangle \langle 2|a+1|\eta \rangle \langle \hat{b}|a+1|\eta \rangle [a1][a\eta]} \\
 & - \frac{\langle a2 \rangle^4 [3\eta]^3}{\langle 12 \rangle \langle a1 \rangle \langle \hat{a}\hat{b} \rangle \langle \hat{a}\hat{b} \rangle \langle 2|3+b|\eta \rangle \langle \hat{a}|3+b|\eta \rangle [3b][b\eta]}
 \end{aligned}$$

$$\text{Ant}(\hat{a}^+, \hat{b}^+ \leftarrow a^+, 1^-, 2^-, 3^-, b^+) =$$

$$\begin{aligned}
 & - \frac{\langle 23 \rangle^3 \langle 1|2+3+b|\eta \rangle^3}{s_{b23} \langle 3b \rangle \langle a1 \rangle \langle \hat{a}\hat{b} \rangle \langle \hat{a}\hat{b} \rangle \langle 2|3+b|\eta \rangle \langle b|2+3|\eta \rangle \langle \hat{a}|2+3+b|\eta \rangle} \\
 & - \frac{\langle 12 \rangle^3 \langle 3|a+1+2|\eta \rangle^3}{s_{a12} \langle 3b \rangle \langle a1 \rangle \langle \hat{a}\hat{b} \rangle \langle \hat{a}\hat{b} \rangle \langle 2|a+1|\eta \rangle \langle a|1+2|\eta \rangle \langle \hat{b}|a+1+2|\eta \rangle} \\
 & + \frac{\langle 3|1+2|\eta \rangle^3}{\langle 3b \rangle \langle \hat{a}\hat{b} \rangle \langle \hat{a}\hat{b} \rangle \langle \hat{a}\hat{b} \rangle \langle a|1+2|\eta \rangle [12][1\eta][2\eta]}
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{\langle 1|2+3|\eta\rangle^3}{\langle a1\rangle\langle\hat{a}\hat{b}\rangle\langle\hat{a}\hat{b}\rangle\langle\hat{a}\hat{b}\rangle\langle b|2+3|\eta\rangle[23][2\eta][3\eta]} \\
 & - \frac{\langle 23\rangle^3[a\eta]^3}{\langle 3b\rangle\langle\hat{a}\hat{b}\rangle\langle\hat{a}\hat{b}\rangle\langle 2|a+1|\eta\rangle\langle\hat{b}|a+1|\eta\rangle[1\eta][a1]} \\
 & - \frac{\langle 12\rangle^3[b\eta]^3}{\langle a1\rangle\langle\hat{a}\hat{b}\rangle\langle\hat{a}\hat{b}\rangle\langle 2|3+b|\eta\rangle\langle\hat{a}|3+b|\eta\rangle[3b][3\eta]}
 \end{aligned}$$

$$\text{Ant}(\hat{a}^+, \hat{b}^+ \leftarrow a^+, 1^-, 2^-, 3^+, b^-) = -\text{Ant}(\hat{b}^+, \hat{a}^+ \leftarrow b^-, 3^+, 2^-, 1^-, a^+)$$

$$\text{Ant}(\hat{a}^+, \hat{b}^+ \leftarrow a^+, 1^-, 2^+, 3^-, b^-) = -\text{Ant}(\hat{b}^+, \hat{a}^+ \leftarrow b^-, 3^-, 2^+, 1^-, a^+)$$

$$\text{Ant}(\hat{a}^+, \hat{b}^+ \leftarrow a^+, 1^+, 2^-, 3^-, b^-) = -\text{Ant}(\hat{b}^+, \hat{a}^+ \leftarrow b^-, 3^-, 2^-, 1^+, a^+)$$

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